Nonrelativistic Diffeomorphism Invariance And Its Applications

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List of publications

1. *"A new formulation of non-relativistic diffeomorphism invariance,"*
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5. *"Gravitational anomalies on the Newton-Cartan background,"*
   Arpita Mitra, Karan Fernandes
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   This thesis is based on the above mentioned papers.

Nonrelativistic Diffeomorphism Invariance And Its Applications

Contents

Chapter 1

Introduction

Diffeomorphisms in non-relativistic systems has gained a renewed importance due to major applications in the field of condensed matter physics, such as in the descriptions of the fractional quantum Hall effect (FQHE), trapped electron gas, and various transport phenomena, to name a few [1–8]. This was initiated in [1], where the role of non-relativistic diffeomorphism invariance (NRDI) to analyze the motion of two dimensional trapped electrons in the context of the FQHE was discussed. This work was inspired by the fact that at low temperatures a Fermi gas behaves as a superfluid. Experimental studies suggest that for the two component Fermi gas an interesting regime exists between Bardeen-Cooper-Schrieffer (BCS) and Bose-Einstein condensate (BEC) known as the "unitary Fermi gas". The relevant effective field theories used to describe this regime involve some variant of the Schrödinger theory on a (2+1)d manifold with universal time. Interestingly, within this effective field theory description [1]- [6] the fields were found minimally coupled to the Newton-Cartan (NC) geometry [3], [5] which provide a covariant description of Newtonian gravity.

The main motivation of this thesis is to investigate diffeomorphism invariance of non-relativistic matter fields. Covariance in non-relativistic physics is subtle due to the absolute nature of time i.e. does not depend on space. NRDI thus has certain distinct features that sets it apart from usual (i.e. relativistic) diffeomorphism invariance. We proposed a field theoretic method to attain NRDI in our work [9, 10], wherein the localization of the Galilean symmetry for field theories in flat spacetime was carried out. The geometrical interpretation of the resulting theory provides a dynamical construction of the NC spacetime as the most general Galilean invariant curved background [11]. We will address this formalism as the "Galilean Gauge theory"(GGT) inspired by the "Poincaré gauge theory" (PGT) [12]. Our method also incorporates torsion in the connection in a straightforward manner. In the process we formulate a massive field theory minimally coupled to the NC background. Local Galilean invariance would be manifest for these fields on this background, similar to the role of Lorentz invariance in the relativistic case. It may appear that a suitable construction of diffeomorphism invariant non-relativistic theories can be obtained from relativistic theories through contraction (light-cone reduction, Wigner-Inonu transformation). While the resulting connection of the manifold would be decomposed into an inertial part and a non-inertial part, in the absence of special boundary conditions this decomposition will not be unique [13]. In addition, the dynamics of fields coupled to the background, particularly to the gauge field contained in the non-inertial part of the connection, are not trivial consequences of known non-relativistic contractions.

In relativistic theories, PGT provides a derivation of matter fields minimally coupled to curved backgrounds through the localization of spacetime symmetries of the fields in flat spacetime [14–16]. The localization procedure for a matter theory invariant under global Poincaré transformations involves promoting the parameters of the transformation to local functions of space and time. The invariance of the theory is broken upon localization. To restore the invariance, compensating fields are introduced in the process by defining covariant derivatives [12]. A very important aspect of this approach is the correspondence of these new fields with the vierbeins and spin-connection of the Riemann-Cartan spacetime. The resulting theory identifies local Poincaré transformations as diffeomorphisms in Riemann-Cartan spacetime. The key difference of this localization method when applied to Galilean invariant field theories will be the nature of the vierbeins. These will differ with the relativistic case not only due to the absolute nature of time, but also on account of the degenerate metrics which they provide a map for. For the Riemann-Cartan spacetime, the vierbein formulation is related directly with the metric formulation because the spacetime manifold is endowed with a nondegenerate metric. In the case of Galilean space and universal time there is no such structure.

In this context it is useful to recall that, following the footsteps of Einstein gravity, a covariant geometrical formulation of Newtonian gravity was worked out by Elie Cartan [17] and subsequently developed in other works [13, 18–23]. This construction is well known as the NC geometry in the literature and helps in appreciating Newtonian gravity as a non-relativistic limit of General Relativity. In Cartan's viewpoint, universal time was regarded as a scalar function. The trajectories of neutral test particles can be viewed as geodesics in curved spacetime. The curved background has to be invariant under Galilean transformations. Each space slice at constant time is flat and endowed with a three dimensional metric with an orthonormal coordinate basis. This implies that the parallel transport of a vector around a closed curve entirely in space will return it to its initial position. However, if we consider the transport forward in time followed by a spatial one and then trace it back by a temporal and a following spatial transport, it can be observed that the vector will not return to its original position. Thus geodesics along a spatial slice that are initially parallel remain always parallel but initially parallel geodesics of spacetime will get pushed away by spacetime curvature [24].

We will now briefly discuss other approaches used in the derivation of minimal coupling to, and the geometry of, curved non-relativistic backgrounds. This will serve to place our work in a clear context with respect to these approaches. The minimal gravitational coupling of the Newtonian theory had been initially considered in [25, 26], whose results have been reviewed in [8]. In [27] it was demonstrated that the 4-d NC geometry can also be formulated on a 5-d spacetime through Bargmann lifting. Using this method one can avoid the degenerate metric structure of the NC geometry and can formulate a corresponding action principle. It gained renewed attention in [1] where the minimal coupling of non-relativistic particles (electrons) to the external gauge field and the metric were determined by using principles of effective field theory. The invariance of the derived action under time dependent diffeomorphisms had been restored by demanding non-canonical transformations of the spatial external gauge fields, which leads to problems when considering the flat space limit [9, 28]. In this limit, the flat space Galilean transformations are restored through a specific assumption, involving a particular relation between the gauge parameter and the boost parameter. In contrast, the flat limit can easily be obtained in our field theoretic approach [9].

Other approaches have been put forward to determine the nature of curved non-relativistic backgrounds directly from the consideration of non-relativistic symmetries. One of these involves the derivation of the background geometry with appropriate metric and curvature tensors, by gauging the centrally extended Galilean algebra (Bargmann algebra) [29]. The conformal extension of this procedure has been carried out in [30]. However, it should be stressed that this is a strictly algebraic approach without reference to any dynamical content of the underlying theory. In addition, the approach necessarily requires the imposition of curvature constraints in order to derive the connection, which formally results in a torsionless theory. Torsion is eventually accounted for in [30] by defining it as the antisymmetric piece of a metric compatible and boost invariant connection, with the further definition of the dilatation gauge field in terms of temporal one-form and its generalized inverse. Yet another approach, which is very closely related to the gauging approach mentioned above is the coset construction [7, 31, 32]. Given a particular symmetry group, and through a prudent choice of a subgroup within it, a coset can be defined which determines the background geometry invariant under the symmetry group. The main feature of this approach is that different choices of the subgroup can lead to several possible realizations of non-relativistic curved backgrounds [31]. The general spacetime connection follows directly from the construction of the Maurer-Cartan form within the coset formalism.

Central to the success of these approaches, as well as our own, are the presence of vierbeins. The coset construction for any non-local symmetry group necessarily involves the use of vierbeins. The same holds true for gauging the algebra directly as in [30], and the localization of symmetries in our work [9, 11]. In these works, either implicitly or explicitly, the flat space theory corresponds to the tangent frame on which the given theory is invariant under global spacetime transformations. The vierbeins serve to map the theory to the appropriate curved spacetime, which in turn renders the theory invariant under local spacetime transformations. Through their involvement, the end result is guaranteed to be manifestly covariant and independent of any specific choice of coordinates. In the context of the coset construction, this statement corresponds to a gauge fixing choice of the parameters [31]. Vierbeins are also central to the description in non-relativistic theories. In relativistic theories diffeomorphism invariance implies the invariance of any theory under general coordinate transformations. In non-relativistic physics it involves a mixing of actual diffeomorphisms with tangent space transformations that act on the vierbeins of the geometry.

In contrast to the approaches described in previous paragraphs, localizing the spacetime symmetries has two specific advantages. The first is that it can study NRDI and determine the minimal coupling of any non-relativistic field theory (scalar, vector, etc.) to the corresponding curved background by its direct involvement from the onset. The second feature is that it reveals that the vierbeins and the relations between them, as well as the form of the connection, are as much a result of the generators being considered as they are of the dependence on the coordinates used at the time of localization. In particular, bearing the non-relativistic nature of absolute time, the parameters of temporal transformations depend only on time and not space. This in turn affects which vierbeins do result from the procedure and serves to elucidate the relation between the vierbeins and the localization of the parameters one begins with.

In general, non-relativistic many particle systems are described by tensor products of the Hilbert space and a many particle Hamiltonian on the product space. An alternative formalism is second quantization. This can be reinterpreted as non-relativistic field theory (NRFT) which can be regarded as low energy effective theories of relativistic systems. When the momentum is much less than the rest mass, NRFT is expected to be a good description of the physics. However, for momentum greater than rest mass, ultraviolet divergences arise in the NRFT. These field theories have to be invariant under the Galilean transformations and unlike the relativistic case, particle number is always conserved here. Galilean transformations involve a time translation, spatial translation, spatial rotation and a boost. The Galilean algebra can be further extended to include a mass operator as a Casimir invariant. This whole algebra is known as the Bargmann algebra (centrally extended Galilean algebra) [33]. Other symmetries, such as gauge invariance and conformal invariance can also be included. Non-relativistic conformal field theories can be 'massive' as mass is a passive parameter in such theories.

Scale transformations in non-relativistic systems are in general anisotropic due to the unequal footing of space and time [34]. One well known non-relativistic scale transformation is 'Lifshitz scaling'. In this case time gets rescaled 'z' times as compared to the space coordinates, where 'z' is called the dynamical exponent. This scaling plays an important role in strongly coupled systems. Holographic investigations have demonstrated that they are relevant in the description of strange metals [35]. It is also relevant in the description of the FQHE [36], the Aharanov-Bohm effect [37], as well as the temperature dependence of transport coefficients in the hydrodynamic description of condensed matter systems with ordinary critical points [38]. Motivated by these observations we will also investigate scale invariant non-relativistic field theories on curved backgrounds. The Bargmann algebra with $z = 2$ Lifshitz scaling and special conformal transformations is known as the Schrödinger algebra. This algebra can be viewed as a non-relativistic extension of the relativistic conformal algebra.

Many continuum non-relativistic field theories admit a fluid description and are expected to be realised in low energy physics experiments. Thus the covariant description of such fluids will be relevant in several condensed matter applications. The recent literature on the NC background in large part addresses some of these topics. A fluid is characterized by its conservation and continuity equations. In considering the NC background, one finds that stress and energy are separately conserved, as is expected from a non-relativistic field theory. The additional non-inertial part of the connection provides corrections to the usual stress tensor of flat space [25], [39]. In considering the scale invariant extension of the NC background, further corrections result due to the gauge field related to non-relativistic dilations [40]. However, the laws of thermodynamics are not affected, and one of the main applications in our treatment of fluids is to demonstrate that the second law continues to hold. Exploring the modification of transport properties requires the consideration of specific systems, for which we have considered the effective field theory of the Quantum Hall fluid. This is described through the Landau-Ginzburg model, which involves the Schrödinger field minimally coupled to a background electromagnetic field as well as a statistical gauge field [41–43]. Coupling with the spin connection of the curved background follows from considering the Schrödinger field as a composite boson, which was first demonstrated by Wen and Zee [44]. This coupling does modify the usual transport relation of the stress-energy tensor in the fractional Quantum Hall effect. In considering the scale invariant NC background, we have demonstrated that there is in addition to the usual Hall viscosity an additional term which can be interpreted as an expansion. This follows from the lowest order correction to the effective field theory due to the scale invariant NC background. Clearly a more detailed investigation into the quantization of the composite boson model and its loop effects promise to be even more interesting.

The NC background can have additional consequences related to the quantization of non-relativistic field theories. Classical dynamics deals with tree level processes of a quantum field theory. One of the simplest quantum corrections which occur at one loop involves the presence of anomalies. Anomalies arise due to the consideration of quantum fields in the presence of external gravitational or gauge fields and represent the failure of classically conserved currents to hold at the quantum level. We have derived the trace and diffeomorphism anomalies of the Schrödinger field minimally coupled to the NC background using Fujikawa's path integral approach [45]. This approach enables us to determine the anomaly corresponding to a classical symmetry transformation by evaluating the regulated trace of the Jacobian for the quantized fields. This trace is evaluated for the Schrödinger fields on the torsion-free NC background using a non-relativistic plane wave basis. We find that in $2+1$ dimensions the trace anomaly contains terms which have a form similar to that of the $1+1$ and $3+1$ dimensional relativistic anomalies. This result demonstrates that the NC background which satisfies the Frobenius condition possesses a Type A trace anomaly, in contrast with the result of Lishitz spacetimes. Through the derivation, we also determine the coefficients and demonstrate that gravitational anomalies for this theory always arise in odd dimensions [46]. This is in contrast with relativistic theories which always arise in an even number of dimensions [47]. We further demonstrate that the coefficient of the term similar to the $1+1$ dimensional relativistic anomaly satisfies a c-theorem condition. This allows us to further investigate the RG flow of certain systems on the NC background, part of which was initiated in [99]. Such investigations will be relevant in understanding the response of condensed matter systems to spatial stresses and deformations.

1.1 Outline of the thesis

I will now briefly describe the outline of my thesis

- *Chapter 2* considers a detailed discussion on the different approaches to Poincaré gauge theory, beginning with the Lie algebraic approach. The connection between Lie algebraic and field theoretic approaches has been highlighted to better understand the Galilean Gauge theory (GGT) formulated in the following chapter. The field theoretic approach has been demonstrated through the example of the complex Klein-Gordon field. In the last subsection, the geometrical interpretation of the local Poincaré invariant action is discussed. This will help in appreciating the construction of the NC geometry from the localization of Galilean transformations in chapter 5.

- *Chapter 3* begins with a brief description of Schrödinger fields and the Bargmann group. Following this, we present our proposed formalism for localizing the Galilean symmetry of non-relativistic scalar fields. This localization procedure results in a local Galilean invariant scalar action. New gauge fields which were introduced during localization will be identified with the geometrical objects in chapter 5. As an example we consider the complex Schrödinger scalar field in $3+1$ dimensions. As an application of the localization procedure we also achieve a spatial diffeomorphism invariant action from the local one by introducing a spatial metric.

- *Chapter 4* demonstrates that our formalism can be easily generalized to couple vector fields to non-relativistic curved backgrounds. We begin by considering a complex scalar field interacting with an external gauge field in flat space. By localising the corresponding spacetime symmetries we formulate this theory in curved space. For time dependent diffeomorphisms our theory predicts the appearance of a new auxiliary field which has no kinetic term in the action. It can be considered as an external field acting on the electron which originates due to the curved background. As an example of a dynamical gauge field we consider the Chern-Simons theory. In contrast to the literature where it has been reported that the Chern-Simons term obstructs the formulation of NRDI, we have successfully derived the Chern-Simons action on non-relativistic curved backgrounds. In addition we have demonstrated that the original gauge symmetry of the model is unaffected by the localization procedure.

- *Chapter 5* begins with a review of the basic properties of the NC geometry which have been derived over the years. A comparison of the results with the ADM decomposition of general relativity is briefly discussed. An analysis of the Lie derivatives acting on the metrics and the NC gauge field has also been provided. Much of these properties will be essential in understanding the dynamics of fields considered on this background, covered in subsequent chapters. In the following section a dynamical construction of the NC background has been provided using the results of GGT from the earlier chapters. The fields introduced during the localization procedure are used to derive the structures of the NC geometry.

- *Chapter 6* includes the anisotropic scale transformation in the localization procedure. Here we demonstrate that the resulting scale covariant NC background involves an additional gauge field related to these dilatations. We further exhibit how this background admits a Weyl tensor analogous to that of relativistic backgrounds. The construction of this section will be necessary in understanding the dynamics of fields on the scale covariant NC background, which is discussed in the next chapter.

- *Chapter 7* reviews basic properties and applications of fluids on non-relativistic curved backgrounds. First, we review the covariant description of fluids on the usual NC background - the expressions for its shear, acceleration, expansion and vorticity, as well as the stress tensor and currents. These relations and expressions are then considered on the scale covariant NC background through the construction of a Weyl covariant formalism. As an application I consider the Landau-Ginzburg effective model for the FQHE. I demonstrate that there exists a response function related to the expansion of the Hall droplet which results due to perturbations of the spatial metric.

- *Chapter 8* considers the trace and diffeomorphism gravitational anomalies resulting from the Schrödinger field on the NC background. It is shown using Fujikawa's approach that the trace anomaly of the Schrödinger field in $2+1$ dimensions involves a result with two parts - one which takes the form of the $3+1$ dimensional relativistic trace anomaly and another which is of the form of the $1+1$ dimensional anomaly. It is the latter piece which is shown to satisfy a c-theorem condition through the local RG formalism and the Wess-Zumino consistency condition. The result for the diffeomorphism anomaly further demonstrates that the trace and diffeomorphism anomalies for the Schrödinger field in $2+1$ dimensions share analogous relations with those of the scalar field in $1+1$ dimensional relativistic backgrounds.

- *Chapter 9* contains the conclusions and future directions.

Chapter 2

Localization of Poincaré symmetry

Poincaré gauge theory (PGT) is an alternative approach to Gravitation theory pioneered by Utiyama [14], Kibble [15] and Sciama [16]. The idea was to localize the corresponding spacetime symmetry- *Poincaré symmetry* of a field theory in Minkowski spacetime. The importance of this procedure is that gauging the Poincaré symmetry in Minkowski spacetime results in a diffeomorphism (diff) invariant theory in the Riemann-Cartan spacetime. The global Poincaré transformation in global Cartesian coordinates in Minkowski space is,

$$x^\mu \rightarrow x^\mu + \epsilon^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \xi^\mu \tag{2.1}$$

where ϵ^μ is the translation parameter and $\omega^\mu{}_\nu$ is the rotation parameter. These parameters are constants as the corresponding transformations are global. In the Utiyama procedure, the following step is to consider the local version of Poincaré transformations where the parameters will now depend on spacetime. However, following localization the rotation part is no longer independent since we can write the local transformations as,

$$x^\mu \rightarrow x^\mu + \xi^\mu(x), \quad \xi^\mu(x) = \epsilon^\mu(x) + \omega^\mu{}_\nu(x) x^\nu \tag{2.2}$$

This feature in Utiyama's approach was later revisited and successfully interpreted. To begin with Utiyama's approach one has to start with a field theory in Minkowski space,

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.3}$$

In the following step one has to implement an active Poincaré transformation on the fields due to the shortcoming of local passive Poincaré transformations Eq. (2.2) described in the previous paragraph. This implies that we replace the original fields by other fields which have been rotated and translated with respect to their former ones. In addition, one has to introduce a coordinate independent coframe. In Minkowski space the tetrad bases (e_α) and the Cartesian coordinate bases are related to each other via

$$e_\alpha = \delta^i_\alpha e_i, \quad e^\alpha = \delta^\alpha_i e^i \tag{2.4}$$

The fields transform under the active Poincaré transformation in the following manner,

$$\phi(x) \to \tilde{\phi}(x) = \left[1 + \tilde{\omega}^{\alpha\beta} M_{\alpha\beta} - \tilde{\epsilon}^\alpha \partial_\alpha\right] \phi(x) \tag{2.5}$$

where $\tilde{\omega}^{\alpha\beta} = \omega^{\alpha\beta}$ and $\tilde{\epsilon}^\alpha = \epsilon^\alpha + \omega_\beta{}^\alpha \delta^\beta_i x^i$. Therefore the translation part ($\tilde{\epsilon}^\alpha$) in Eq. (2.5) consists of the original translation part of Eq. (2.1) and a rotation induced translation. In this set up, during localization, the rotation part retains its independent character and the matter fields are described with respect to the tetrad frame.

Upon localization the invariance of the action Eq. (2.3) is lost. To restore invariance, gauge potentials corresponding to the translation and rotation have to be introduced. Translational gauge potentials are identified with the tetrads. Rotational gauge fields are included through the definition of covariant derivatives and can be interpreted as the connection of the background. Inhomogeneous transformations of these gauge potentials will ensure the local Poincaré invariance of the theory. The field strengths corresponding to the translation and rotation can be identified geometrically with the torsion and Riemann-Cartan curvature respectively.

The drawback of Utiyama's approach to identify the diffeomorphism parameter as a combination of an independent translation parameter and an independent rotation parameter inspired a different approach to PGT. This approach is algebraic and concerns itself with gauging the Poincaré group directly. This is similar to the procedure introduced by Stelle and West [48] for the $SO(3,2)$ group spontaneously broken to the Lorentz group. In the group gauging framework, one considers the Poincare gauge theory similar to any ordinary nonabelian gauge theory, without discarding the translation part of the Poincaré symmetry in favour of general coordinate transformations. However, the translation part of the transformation does not allow the Poincaré group to have a pure Yang-Mills type gauge description. In the next subsection we will briefly discuss the group gauging procedure.

2.1 Lie algebraic approach to PGT

The Poincaré group is a composition of translation and Lorentz generators. The group has ten parameters. Four of them correspond to translations (ϵ^μ) and six of them correspond to Lorentz transformations ($\omega^{\mu\nu}$). The Lorentz parameters and generators are antisymmetric. We denote the generators of translation and Lorentz transformations as P_μ and $M_{\mu\nu}$ respectively. These generators satisfy the following commutation relations,

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, P_\sigma] &= \eta_{\mu\sigma} P_\nu - \eta_{\nu\sigma} P_\mu \\
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\nu\rho} M_{\mu\sigma}
\end{aligned} \tag{2.6}$$

The parameters $\epsilon^\mu, \omega^{\mu\nu}$ are constant for the global Poincaré group. The global symmetry is transformed to a local one if the parameters are considered as functions of spacetime. To restore the algebra Eq. (2.6) new gauge fields are introduced. The transformations of these new gauge fields are derived within the framework of nonabelian gauge theories. Different techniques have been proposed to find the connection between the diffeomorphism and translation parameters. We will try to highlight the main results of this Lie algebraic approach.

A Lie algebra valued gauge potential can be introduced in the following manner,

$$A_\mu = P_a e_\mu{}^a + \frac{1}{2} M_{ab} \varpi_\mu^{ab} \tag{2.7}$$

where 'a' represent local indices while 'μ' represent global ones. The introduced gauge fields $e_\mu{}^a$ and ϖ_μ^{ab} are associated with translations and Lorentz transformations respectively. They will be identified as the vierbein and spin connection of Riemann-Cartan spacetime. The gauge transformation for the potential A_μ is given by

$$\delta A_\mu = D_\mu \Lambda = \partial_\mu \Lambda + [A_\mu, \Lambda] \tag{2.8}$$

where 'Λ' is the gauge parameter and 'D_μ' is the covariant derivative. The gauge parameter can be expressed in terms of the Poincaré group parameters and generators as,

$$\Lambda = \epsilon^\alpha P_\alpha + \frac{1}{2} \omega^{\alpha\beta} M_{\alpha\beta} \tag{2.9}$$

where ϵ^α and $\omega^{\alpha\beta}$ are now functions of spacetime. Using Eq. (2.7), Eq. (2.8) and Eq. (2.9) and by exploiting the Poincaré algebra Eq. (2.6), we obtain the following transformation rules for $e_\mu{}^a$ and ϖ_μ^{ab},

$$\begin{aligned}
\delta e_\mu^a &= \partial_\mu \epsilon^a - \varpi_\mu{}^a{}_b \epsilon^b + \omega^a{}_b e_\mu^b \\
\delta \varpi_\mu^{ab} &= \partial_\mu \omega^{ab} + \omega^a{}_e \varpi_\mu^{eb} + \omega^b{}_e \varpi_\mu^{ae}
\end{aligned} \tag{2.10}$$

The field strength $F_{\mu\nu}$ is given by its usual definition

$$\begin{aligned}
F_{\mu\nu} &= [D_\mu, D_\nu] \\
&= P_a F^a{}_{\mu\nu} + \frac{1}{2} M_{ab} F^{ab}{}_{\mu\nu}
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
F_{\mu\nu}{}^a &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a - \varpi_\mu{}^a{}_c e_\nu{}^c + \varpi_\nu{}^a{}_c e_\mu{}^c \\
F^{ab}{}_{\mu\nu} &= \partial_\mu \varpi^{ab}{}_\nu - \partial_\nu \varpi^{ab}{}_\mu - \varpi_\mu{}^a{}_c \varpi^c{}_\nu{}^b + \varpi_\nu{}^a{}_c \varpi^c{}_\mu{}^b
\end{aligned} \tag{2.12}$$

The field strength transforms covariantly under the nonabelian gauge transformation.

The main aim of this gauging procedure is to relate the transformations Eq. (2.10) with appropriate spacetime transformations. Analysing Eq. (2.10) it can be realized that the variation of e^a_μ is determined by both translation and rotations whereas the transformation of ϖ^{ab}_μ is entirely determined by the local Lorentz rotations. This suggests that diffeomorphisms can be connected to the translation parameter ϵ^a in this Lie algebraic approach. To understand this connection we define the diffeomorphism parameter,

$$\xi^\lambda = e^\lambda{}_a \epsilon^a \tag{2.13}$$

where $e^\lambda{}_a$ is the inverse of e^a_μ following,

$$e_\lambda{}^d e^\lambda_e = \delta^d_e \quad ; \quad e_\lambda{}^d e^\mu_d = \delta^\mu_\lambda \tag{2.14}$$

The vierbein helps to express any vector or tensor in the flat Minkowski spacetime to that of the curved spacetime. However we still have to show that it satisfies the correct transformation rules under general coordinate transformations.

To study the dependence of $\varpi_\mu{}^a{}_b$ on $e_\lambda{}^d$ we have to impose a curvature constraint,

$$F_{\mu\nu}{}^a = 0 \tag{2.15}$$

Eq. (2.15) and Eq. (2.12) together imply,

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \varpi_\mu{}^a{}_c e_\nu{}^c + \varpi_\nu{}^a{}_c e_\mu{}^c = 0 \tag{2.16}$$

To get an expression of ϖ^{ab}_μ in terms of $e_\mu{}^a$ we contract Eq. (2.16) by $e^\mu_d e^\nu_b$,

$$0 = e^\mu_d e^\nu_b \partial_\mu e^a_\nu - e^\mu_d e^\nu_b \partial_\nu e^a_\mu - \varpi_\mu{}^a{}_b e^\mu_d + \varpi_\mu{}^a{}_d e^\mu_b \tag{2.17}$$

Changing d, b and a cyclically will provide the following two expressions,

$$0 = e^\mu_b e^\nu_a \partial_\mu e^d_\nu - e^\mu_b e^\nu_a \partial_\nu e^d_\mu - \varpi_\mu{}^d{}_a e^\mu_b + \varpi_\mu{}^d{}_b e^\mu_a \tag{2.18}$$

and

$$0 = e^\mu_a e^\nu_d \partial_\mu e^b_\nu - e^\mu_a e^\nu_d \partial_\nu e^b_\mu - \varpi_\mu{}^b{}_d e^\mu_a + \varpi_\mu{}^b{}_a e^\mu_d \tag{2.19}$$

Now by adding Eq. (2.17) and Eq. (2.18) , and subtracting Eq. (2.19) from the sum, we obtain,

$$\begin{aligned}\varpi_\mu{}^{ab} = \frac{1}{2}\Big[& -e^{\lambda a}\left(\partial_\mu e^b_\lambda - \partial_\lambda e^b_\mu\right) + e^{\lambda b}\left(\partial_\mu e^a_\lambda - \partial_\lambda e^a_\mu\right) \\ & + e^c_\mu e^{\lambda a} e^{\rho b}\left(\partial_\lambda e^c_\rho - \partial_\rho e^c_\lambda\right)\Big]\end{aligned} \tag{2.20}$$

The next step will be to verify that this expression of $\varpi_\mu{}^{ab}$ Eq. (2.20) is consistent with the variation of e^a_μ under a diffeomorphism. To do this one has to substitute

ϖ_μ^{ab} in $\delta e_\mu{}^a$ by Eq. (2.20). First we will simplify the second term of the right hand side of the first equation of Eq. (2.10),

$$\varpi_\mu{}^{ab}\epsilon^b = \partial_\mu\zeta^a - \xi^\lambda\partial_\lambda e_\mu{}^a - \partial_\mu\xi^\lambda e_\lambda{}^a + \frac{1}{2}\Big[\xi^\lambda\partial_\mu e_\lambda{}^a - \xi^\lambda\partial_\lambda e^a_\mu$$
$$+\epsilon^b e^{\lambda b}\partial_\mu e_\lambda{}^a + \epsilon^b e^{\lambda a}\partial_\lambda e_\mu{}^b + \epsilon^b e_\mu{}^c\partial_\rho e_\lambda{}^c(e^{\rho a}e^{\lambda b} - e^{\lambda a}e^{\rho b})\Big] \tag{2.21}$$

Substituting Eq. (2.21) in Eq. (2.10) gives,

$$\delta e^a_\mu = \xi^\lambda\partial_\lambda e_\mu{}^a + \partial_\mu\xi^\lambda e_\lambda{}^a + \omega^{ab}e^b_\mu + \frac{\epsilon^b}{2}\Big[\Big(e^{\lambda a}\partial_\lambda e_\mu{}^b + e^b_\lambda\partial_\mu e^{\lambda a}\Big)$$
$$+\Big(e^{\lambda b}\partial_\lambda e^a_\mu + e^a_\lambda\partial_\mu e^{\lambda b}\Big) - e^c_\mu e^{\lambda a}\Big(e^{\rho b}\partial_\rho e_\lambda{}^c + e_\rho{}^c\partial_\lambda e^{\rho b}\Big)\Big] \tag{2.22}$$

The expected transformation of e^a_μ will be obtained provided the term in the parenthesis vanish. However, this term does not vanish by algebraic means. One way to achieve this is to invoke a flat geometry in the tangent space and introduce the basis vectors $e_{(a)}$ along with the basis one forms $\omega^{(a)}$. Then the Lie derivative of $\omega^{(a)}$ along $e_{(b)}$ must vanish. Thus, in this coordinate basis [49]

$$e^{\lambda a}\partial_\lambda e_\mu{}^b + e_\lambda{}^b\partial_\mu e^{\lambda a} = 0 \tag{2.23}$$

This relation Eq. (2.23) ensures that e^a_μ transforms correctly under diffeomorphisms. Thus the Lie algebraic approach helps in identifying the Poincaré group in Minkowski spacetime with the tangent space at a point in the curved spacetime.

Note that one shortcoming of this approach is that setting the gauge curvature to zero was not sufficient to ensure the identification of the translation gauge field with the vierbein. This required an additional geometrical input. In contrast, one directly finds this connection between the translation gauge field and the vierbein in Utiyama's approach. The connection between the Utiyama approach and the Lie algebraic approach has been established in [48] by introducing an extra Poincaré translation vector. In the following section we will briefly discuss this aspect.

2.2 Connection between Lie algebraic and field theoretic approach to PGT

In this section we will highlight the connection between the algebraic and field theoretic approaches following [48]. According to [48] to construct a gauge theory of the Poincaré group, nondynamical gauge degrees of freedom q^a have to be introduced in addition to the gauge potentials. q^a will transform under the infinitesimal global Poincaré transformations,

$$\delta q^a = \omega^a{}_b q^b + \epsilon^a \tag{2.24}$$

If we consider local transformations where the parameters are functions of spacetime, then an action invariant under the global transformation Eq. (2.24) would no longer be invariant. In order to restore the invariance, the ordinary derivatives have to be replaced by suitable covariant derivatives. Two types of covariant derivatives can be introduced. One is $D_\mu q^a$, such that it transforms inhomogeneously like q^a.

$$D_\mu q^a = \partial_\mu q^a + Q^{ab}{}_\mu q_b \tag{2.25}$$

This new derivative in Eq. (2.25) will transform as a covariant derivative on Eq. (2.24) should,

$$\delta(D_\mu q^a) = \omega^a{}_b D_\mu q^b + D_\mu \epsilon^a \tag{2.26}$$

Eq. (2.24), Eq. (2.25) and Eq. (2.26) enforces the transformation of $Q^{ab}{}_\mu$,

$$\delta Q^{ab}{}_\mu = \partial_\mu \omega^{ab} + \omega^a{}_e Q^{eb}{}_\mu + \omega^b{}_e Q^{ae}{}_\mu \tag{2.27}$$

It is clear that the new field $Q^{ab}{}_\mu$ transforms like $\varpi^{ab}{}_\mu$ introduced in the last section. Thus $Q^{ab}{}_\mu$ can be identified with $\varpi^{ab}{}_\mu$.

The second kind of covariant derivative can be introduced to restore the invariance of the action under local transformations provided the covariant derivative transforms homogeneously. We will define this derivative as,

$$\mathcal{D}_\mu q^a = D_\mu q^a + Q^a_\mu \tag{2.28}$$

We demand that this derivative will transform like,

$$\delta(\mathcal{D}_\mu q^a) = \omega^a{}_b \mathcal{D}_\mu q^b \tag{2.29}$$

where $Q^a{}_\mu$ is the new gauge field. The condition Eq. (2.29) ensures the following transformation of $Q^a{}_\mu$,

$$\delta Q^a{}_\mu = \partial_\mu \epsilon^a - Q_\mu{}^a{}_b \epsilon^b + \omega^a{}_b Q^b{}_\mu \tag{2.30}$$

Eq. (2.30) implies that the gauge field $Q^a{}_\mu$ can be identified with $e^a{}_\mu$ of the last section. The next step will be same as the Lie algebraic approach - the introduction of a Lie algebra valued gauge potential Eq. (2.7). In the next section a brief analysis of gauging the Poincaré symmetry of a Klein-Gordon field in Minkowski spacetime will be given. This will help in understanding the formalism proposed by us for the non-relativistic case.

2.3 Gauging the Poincaré symmetry for a field theoretic model

In this section we will revisit PGT as a field theoretic approach. This approach can be applied to any Poincaré invariant field theory of interest, defined on Minkowski space. As an example we will here consider the complex Klein-Gordon scalar field. The corresponding action is given by

$$S = \int d^4x \left[\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi\right] \tag{2.31}$$

The action Eq. (2.31) is invariant under Eq. (2.1). It can be demonstrated using the following variation of the action under any coordinate transformation,

$$\begin{aligned}
\Delta S &= \int d^4x' \; \mathcal{L}'(x') - \int d^4x \; \mathcal{L}(x) \\
&= \int d^4x \; [\mathcal{L}'(x') - \mathcal{L}'(x) + \mathcal{L}'(x) - \mathcal{L}(x)] + \int d^4x \; \partial_\mu\xi^\mu\mathcal{L}(x)] \\
&= \int d^4x \; [\xi^\mu\partial_\mu\mathcal{L}(x) + \delta_0\mathcal{L}(x) + \partial_\mu\xi^\mu\mathcal{L}(x)]
\end{aligned} \tag{2.32}$$

The action is invariant when $\Delta S = 0$. This results from $\mathcal{L}$ satisfying,

$$\triangle\mathcal{L} = \xi^\mu\partial_\mu\mathcal{L}(x) + \delta_0\mathcal{L}(x) + \partial_\mu\xi^\mu\mathcal{L}(x) = 0 \tag{2.33}$$

To verify whether the complex scalar field satisfies the condition Eq. (2.33) one has to use the form variation of the field. The form variation is defined as, $\delta_0\phi = \phi'(x,t) - \phi(x,t)$. Note that we are dealing with scalar fields for which $\phi'(x',t') = \phi(x,t)$. Therefore under the transformation Eq. (2.1) the form variation of ϕ is,

$$\begin{aligned}
\delta_0\phi &= -[\omega^\mu{}_\nu x^\nu + \epsilon^\mu]\partial_\mu\phi \\
&= -\left(\frac{1}{2}\omega^{\lambda\nu}M_{\lambda\nu} - \epsilon^\nu P_\nu\right)\phi = \left(-\frac{1}{2}\omega^{\lambda\nu}\Sigma_{\lambda\nu} + \xi^\nu P_\nu\right)\phi
\end{aligned} \tag{2.34}$$

where $\omega^\mu{}_\nu x^\nu + \epsilon^\mu = \xi^\mu$. $M_{\mu\nu}$ and P_μ are rotation and translation generators respectively. $\Sigma_{\mu\nu}$ is the generator corresponding to the spin part in the rotation. The derivative of ϕ will transform as,

$$\begin{aligned}
\delta_0(\partial_\mu\phi) &= -[\omega^\rho{}_\sigma x^\sigma + \epsilon^\rho]\partial_\mu\partial_\rho\phi - \omega^\rho{}_\mu\partial_\rho\phi \\
&= -\frac{1}{2}\omega^{\alpha\beta}\Sigma_{\alpha\beta}\partial_\mu\phi - \xi^\nu\partial_\nu\partial_\mu\phi - \omega^{\lambda\mu}\partial_\lambda\phi
\end{aligned} \tag{2.35}$$

Applying Eq. (2.34), Eq. (2.35) in Eq. (2.33) one can prove that the action Eq. (2.31) satisfies $\triangle\mathcal{L} = 0$. The total variation of the field and the coordinates under localized Poincaré transformations are defined as,

$$\begin{aligned}
\delta\phi &= -\frac{1}{2}\omega^{ij}(x)\Sigma_{ij}\phi \\
\delta x^\mu &= \xi^\mu(x)
\end{aligned} \tag{2.36}$$

where we have labelled the total variation of the field in Latin indices and those of the coordinates in Greek indices.

For global infinitesimal transformations, the parameters are constants. When this symmetry is localised the parameters will depend on spacetime. One can separate coordinate and field transformations by choosing ξ^μ in $\delta x^\mu = \xi^\mu$ as the independent parameter instead of ϵ^μ. This choice allows one to consider generalised transformations with $\xi^\mu = 0$ but having non-zero $\omega^{\mu\nu}$. The action would not be invariant under this local transformation as the transformation of the field derivatives given in Eq. (2.35) now changes to,

$$\delta_0(\partial_k\phi) = -\frac{1}{2}\omega^{ij}\Sigma_{ij}\partial_k\phi - \frac{1}{2}(\partial_k\omega^{ij})\Sigma_{ij}\phi - \xi^\lambda\partial_\lambda\partial_k\phi - \partial_k\xi^\lambda\partial_\lambda\phi \tag{2.37}$$

Therefore to restore the invariance under this local transformation we have to proceed with the following two steps;

- Introduce tetrads (vierbeins) to relate between global and local coordinates
- Introduce some new gauge fields by defining covariant derivatives

The covariant derivative with respect to the global coordinates is,

$$\nabla_\mu\phi = \partial_\mu\phi + \frac{1}{2}\varpi^{ij}_\mu(x)\Sigma_{ij}\phi \tag{2.38}$$

To restore the invariance we require this derivative Eq. (2.38) to transform as,

$$\delta_0(\nabla_\mu\phi) = -\frac{1}{2}\omega^{ij}\Sigma_{ij}\nabla_\mu\phi - (\partial_\mu\xi^\lambda)\nabla_\lambda\phi - \xi^\lambda\partial_\lambda\nabla_\mu\phi \tag{2.39}$$

This enforces the transformation of the new fields ${\varpi^{ij}}_\mu$ as,

$$\delta_0{\varpi^{ij}}_\mu = \partial_\mu\omega^{ij} - (\partial_\mu\xi^\lambda){\varpi^{ij}}_\lambda - \xi^\lambda\partial_\lambda{\varpi^{ij}}_\mu + \omega^{im}{\varpi_m}^j{}_\mu - \omega^{jm}{\varpi_m}^i{}_\mu \tag{2.40}$$

The total variation of $\nabla_\mu\phi$ can be determined from its form variation Eq. (2.39),

$$\delta(\nabla_\mu\phi) = \delta_0(\nabla_\mu\phi) + \xi^\lambda\partial_\lambda\nabla_\mu\phi = -\frac{1}{2}\omega^{ij}\Sigma_{ij}\nabla_\mu\phi - (\partial_\mu\xi^\lambda)\nabla_\lambda\phi \tag{2.41}$$

It is evident that the total variation Eq. (2.41) is not covariant due to the presence of the last term. Therefore we have to define another covariant derivative with respect to local coordinates which will provide the required transformation. This is unlike an ordinary gauge theory where one covariant derivative is sufficient to restore the invariance. This reflects the fact that the theory cannot be described as a pure gauge theory due to the consideration of both translations and Lorentz transformations.

As Eq. (2.38) is not sufficient to restore the local invariance we will define the second covariant derivative as,

$$\nabla_k\phi = {e_k}^\mu\nabla_\mu\phi \tag{2.42}$$

To transform covariantly under local Poincaré transformations this derivative Eq. (2.42) has to vary as,

$$\delta(\nabla_k\phi) = -\frac{1}{2}\omega^{ij}\Sigma_{ij}\nabla_k\phi - {\omega^i}_k\nabla_i\phi \tag{2.43}$$

This will be satisfied provided the new introduced fields ${b_k}^\mu$ transform under the local transformation in the following manner,

$$\delta {e_k}^\mu = {\omega_k}^i {e_i}^\mu + {e_k}^\lambda \partial_\lambda \xi^\mu \tag{2.44}$$

It is easily understood that ${e_k}^\mu$ is the vierbein. After acquiring the required covariant derivatives, the next step is to construct the invariant action. To achieve this we will follow two steps,

- $\mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}'(\phi, \nabla_\alpha \phi)$
- Change the measure to account for $\partial_\mu \xi^\mu \neq 0$ under local transformations

A suitable choice of measure is $e = \det({e^i}_\mu)$ as it satisfies,

$$\delta e + (\partial_\mu \xi^\mu) e = 0 \tag{2.45}$$

which is necessary to satisfy $\Delta \mathcal{L} = 0$ under the local transformation. Thus following the two steps we finally attain an action which preserve its invariance under the local Poincaré transformation.

$$S = \int d^4x \; e\mathcal{L}(\phi, \nabla_k \phi) \tag{2.46}$$

The local Klein-Gordon action will be as follows,

$$S = \int d^4x \; e \left[\eta^{\alpha\beta} {e_\alpha}^\mu \nabla_\mu \phi^* {e_\beta}^\nu \nabla_\nu \phi - m^2 \phi^* \phi \right] \tag{2.47}$$

2.4 Geometrical interpretation of PGT

It is possible to develop a geometric interpretation of the background from both the Lie algebraic and field theoretic approaches to PGT. One has to begin by introducing a metric in terms of the tetrad (vierbein).

$$g_{\mu\nu} = \eta_{\alpha\beta} e^\alpha_\mu e^\beta_\nu, \quad g^{\mu\nu} = \eta^{\alpha\beta} e^\mu_\alpha e^\nu_\beta \tag{2.48}$$

Using Eq. (2.44) we can attain the appropriate transformations of the metric under the diffeomorphism as follows,

$$\delta g_{\mu\nu} = -\partial_\mu \xi^\rho g_{\rho\nu} - \partial_\nu \xi^\rho g_{\mu\rho} - \xi^\rho \partial_\rho g_{\mu\nu}. \tag{2.49}$$

We thus identify the gauge fields ${b^i}_\mu$ introduced during localization as vierbeins. Now we can appreciate the action Eq. (2.47) as the complex Klein-Gordon scalar minimally coupled to a curved background (M),

$$S = \int d^4x \; \sqrt{g} \left[g^{\mu\nu} \nabla_\mu \phi^* \nabla_\nu \phi - m^2 \phi^* \phi \right] \tag{2.50}$$

Next we will derive some relevant quantities corresponding to the background. The two types of covariant derivatives Eq. (2.38), Eq. (2.42) will give rise to two distinct field strengths,

$$[\nabla_\mu, \nabla_\nu]\,\phi = \frac{1}{2}R^{ij}{}_{\mu\nu}\Sigma_{ij}\phi$$
$$[\nabla_k, \nabla_l]\,\phi = \frac{1}{2}R^{ij}{}_{kl}\Sigma_{ij}\phi - T^i{}_{kl}\nabla_i\phi \tag{2.51}$$

where

$$R^{ij}{}_{\mu\nu} = \partial_\mu\varpi^{ij}_\nu - \partial_\nu\varpi^{ij}_\mu + \varpi^i_{l\mu}\varpi^{lj}{}_\nu - \varpi^i_{l\nu}\varpi^{lj}{}_\mu$$
$$T^i{}_{kl} = e_k{}^\mu e_l{}^\nu T^i_{\mu\nu} = e_k{}^\mu e_l{}^\nu(\nabla_\mu e^i_\nu - \nabla_\nu e^i_\mu) \tag{2.52}$$

The quantities $R^{ij}{}_{\mu\nu}$, $T^i_{\mu\nu}$ are called the Lorentz field strength and translational field strength respectively. The Jacobi identities for the commutators results in the following Bianchi identities [12],

$$\epsilon^{\mu\nu\rho\sigma}\nabla_\nu T^i{}_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}R^i{}_{k\rho\sigma}e^k{}_\nu$$
$$\epsilon^{\mu\nu\rho\sigma}\nabla_\rho R^{ij}_{\nu\sigma} = 0 \tag{2.53}$$

If we consider the dynamical curved background and minimally coupled matter fields to it, the free Lagrangian should be an invariant density depending only on the field strengths. Thus the complete Lagrangian [12] is of the following form,

$$\tilde{\mathcal{L}} = e\mathcal{L}_B(R^{ij}{}_{\mu\nu}, T^i_{\mu\nu}) + e\mathcal{L}_M(\phi, \nabla_k\phi) \tag{2.54}$$

where $\mathcal{L}_B$ and $\mathcal{L}_M$ are the Lagrangians corresponding to the dynamical curved background and the matter coupled to it respectively. In contrast to the Yang-Mills theory, in PGT an invariant quantity linear in field derivatives can be constructed,

$$R = e_i{}^\mu e_j{}^\nu R^{ij}{}_{\mu\nu} \tag{2.55}$$

Therefore from Eq. (2.55) one can construct the Einstein-Cartan action [15,16],

$$S_{EC} = \int d^4x\ (-aR + \mathcal{L}_M) \tag{2.56}$$

We can further determine the connection for the curved manifold (M) using the *'vierbein postulate'*. For any general curved background (C) the metric obeys the metricity condition, i.e. vanishing covariant derivative of the metric ($D_\mu(\Gamma)g_{\nu\lambda} = 0$). The manifold (C) is equipped with a linear connection Γ. If the connection is *symmetric* it has the following expression,

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\rho g_{\lambda\nu} + \partial_\nu g_{\lambda\rho} - \partial_\lambda g_{\nu\rho}\right). \tag{2.57}$$

From the commutator of the covariant derivatives the Riemann tensor can be derived,

$$R^\mu{}_{\nu\lambda\rho} = \partial_\lambda\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\lambda} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\lambda}, \tag{2.58}$$

A general connection also has the antisymmetric component called the torsion,

$$T^{\mu}_{\ \lambda\rho} = \Gamma^{\mu}_{\rho\lambda} - \Gamma^{\mu}_{\lambda\rho}. \tag{2.59}$$

Note that the notion of parallel transport in the M-frame works through the spin connection $\varpi^{ij}{}_{\mu}$, while in the C-frame it is described by the manifold connection $\Gamma^{\mu}_{\nu\rho}$. However the two notions will agree as ϖ and Γ are connected through the *'vierbein postulate'*:

$$D_{\mu}(\varpi + \Gamma)e^{i}{}_{\nu} = \partial_{\mu}e^{i}{}_{\nu} + \varpi^{i}_{s\mu}e^{s}{}_{\nu} - \Gamma^{\lambda}_{\nu\mu}e^{i}{}_{\lambda} = 0. \tag{2.60}$$

From Eq. (2.60) one can get Γ in terms of the spin connection ϖ.

$$\Gamma^{\mu}_{\nu\lambda} = e_{i}{}^{\mu}\partial_{\lambda}e^{i}{}_{\nu} + \varpi^{i}{}_{j\lambda}e_{i}{}^{\mu}e^{j}{}_{\nu} \tag{2.61}$$

Deriving the variation of $\Gamma^{\mu}_{\nu\lambda}$ from Eq. (2.61) and comparing with the variation of $\Gamma^{\mu}_{\nu\lambda}$ under diffeomorphism,

$$\delta\Gamma^{\mu}_{\nu\lambda} = -\partial_{\nu}\xi^{\rho}\ \Gamma^{\mu}_{\rho\lambda} - \partial_{\lambda}\xi^{\rho}\ \Gamma^{\mu}_{\nu\rho} + \partial_{\rho}\xi^{\mu}\ \Gamma^{\rho}_{\nu\lambda} - \partial_{\nu}\partial_{\lambda}\xi^{\mu} - \xi^{\rho}\ \partial_{\rho}\Gamma^{\mu}_{\nu\lambda} \tag{2.62}$$

we observe a similar variation of $\delta\varpi^{ij}{}_{\mu}$ in Eq. (2.40). Identifying the tetrad and the spin connection with the 'gauge potentials' $e^{i}{}_{\mu}$ and $\varpi^{ij}{}_{\lambda}$, one can say that spacetime symmetry transformations (namely diffeomorphisms) generate the same transformations as the Poincaré gauge transformations. Using this expression of Γ Eq. (2.61) in the geometric definitions of the Riemann Eq. (2.58) and torsion Eq. (2.59) tensors, we can conclude that [12],

$$\begin{aligned} T^{\mu}{}_{\nu\lambda}(\Gamma) &= e_{i}{}^{\mu}T^{i}{}_{\nu\lambda}(\varpi) \\ R^{\mu}{}_{\nu\lambda\rho}(\Gamma) &= e_{i}{}^{\mu}e_{j\nu}R^{ij}{}_{\lambda\rho} \end{aligned} \tag{2.63}$$

Thus we see that the translational and rotational field strengths $(T^{i}{}_{\nu\lambda}, R^{ij}{}_{\lambda\rho})$ are interpreted as the torsion and the Riemann tensor respectively. In addition, using (Eq. (2.60)) and Eq. (2.48) we recover the *'metricity condition'*:

$$D_{\mu}(\Gamma)g_{\nu\lambda} = D_{\mu}(\varpi + \Gamma)g_{\nu\lambda} = D_{\mu}(\varpi + \Gamma)\eta_{ij}e^{i}{}_{\nu}e^{j}{}_{\lambda} = 0. \tag{2.64}$$

Therefore the correspondence of the Poincaré gauge structure with the geometrical manifold picture has been established. We thus observe that the Poincaré gauge theory lives on a Riemann-Cartan manifold with torsion. In the next chapter we will formulate the localization prescription of Galilean symmetry.

Chapter 3

Localisation of the Galilean symmetry for scalar fields

We begin with a basic description of Schrödinger fields on flat space and its symmetry algebra. Following this a detailed discussion on the localization of the Galilean symmetry for scalar fields is given. As an application the resulting theory is shown to satisfy NRDI.

3.1 Schrödinger fields on flat space

Two fundamental requirements of non-relativistic field theories are that they be Galilean invariant and that the number of particles is a conserved quantity. The conserved particle number is a reflection of the global U(1) symmetry of the theory. Other symmetries like conformal invariance may be imposed to restrict the theory further. A simple example of such a theory is the complex scalar field with the following action,

$$S = \int dt d^3x \left[\frac{i}{2}(\phi^* \partial_t \phi - \phi \partial_t \phi^*) - \frac{1}{2m} \partial_k \phi^* \partial^k \phi \right] \tag{3.1}$$

The first parenthesis of Eq. (3.1) ensures the hermiticity of the action. $\rho = \phi^* \phi$ is the conserved particle number density. The most important feature of this theory is that the kinetic term contains a first order time derivative as well as a second order space derivative. This is sufficient to ensure that ϕ contains an annihilation part and the corresponding particle propagates forward in time. The theory contains no antiparticles by construction. Unlike relativistic theories where mass enters as a coefficient of the potential, here mass is a passive parameter in the kinetic term. The equations of motion for ϕ and ϕ^* following the Euler-Lagrange equation are,

$$
\begin{aligned}
-i\partial_t\phi &= \frac{1}{2m}\Box\phi \\
i\partial_t\phi^* &= \frac{1}{2m}\Box\phi^*
\end{aligned}
\tag{3.2}
$$

The action Eq. (3.1) is invariant under the global gauge transformation,

$$
\phi^{'}(x,t) = e^{i\alpha}\phi(x,t) \tag{3.3}
$$

where α is the gauge parameter. Under this transformation the derivatives acting on the field transform homogeneously. Now if we consider local gauge transformations i.e. the gauge parameter α also depends on spacetime, the action is no longer invariant. To recover the homogeneous transformations of the derivatives on the fields, one has to introduce new covariant derivatives,

$$
\begin{aligned}
D_t\phi &= \partial_t\phi + iA_0\phi \\
D_k\phi &= \partial_k\phi + iA_k\phi
\end{aligned}
\tag{3.4}
$$

These derivatives will transform covariantly provided the gauge fields A_0 and A_i transform under local gauge transformations in the following way,

$$
\begin{aligned}
A_0' &= A_0 - \partial_t\alpha \\
A_i' &= A_i - \partial_i\alpha
\end{aligned}
\tag{3.5}
$$

Replacing partial derivatives by gauge covariant derivatives helps to recover the invariance under local gauge transformations. A_0 and A_i can be identified as scalar and vector potentials respectively in Maxwell's electromagnetism. The application of Noether's theorem results in the conserved current, which for global gauge transformations provide

$$
J_i = \frac{i}{2m}[\phi^*\partial_i\phi - \phi\partial_i\phi^*] \tag{3.6}
$$

Similarly, the conserved current under the local gauge transformation is,

$$
J_i = \frac{i}{2m}[\phi^* D_i\phi - \phi D_i\phi^*] \tag{3.7}
$$

3.2 Bargmann group

Spacetime symmetries of non-relativistic systems can be represented in terms of the Galilean group. The flat spacetime symmetry group corresponds to spatial rotations, Galilean boosts, time and space translations. The generators for rotation, boost, time translation and spatial translation are respectively,

$$
\begin{aligned}
\lambda_{ab} =& x_a\partial_b - x_b\partial_a \\
K_a =& m x_a \\
H =& \partial_t \\
P_a =& \partial_a
\end{aligned}
\tag{3.8}
$$

The Galilean group has 10 parameters and is thus 10 dimensional. Unlike Lorentz transformations every parameter of the Galilean group is real. This group is completely determined by the composition rule. According to the Wigner theorem, for every continuous spacetime transformation there exists an unitary operator which acts on the space of states and observables. The corresponding Lie algebra is spanned by the generators Eq. (3.8), subject to the following commutation relations,

$$\begin{aligned}
[H, P_a] &= 0 \\
[P_a, P_b] &= 0 \\
[\lambda_{ab}, H] &= 0 \\
[K_a, K_b] &= 0 \\
[\lambda_{ab}, \lambda_{cd}] &= \delta_{ac}\lambda_{bd} - \delta_{ad}\lambda_{cb} + \delta_{bd}\lambda_{ac} - \delta_{bc}\lambda_{ad} \\
[\lambda_{ab}, P_c] &= \delta_{ac}P_b - \delta_{bc}P_a \\
[\lambda_{ab}, K_c] &= \delta_{ac}K_b - \delta_{bc}K_a \\
[K_a, H] &= iP_a \\
[K_a, P_b] &= 0
\end{aligned} \tag{3.9}$$

This Lie algebra can be appreciated as the classical limit of the Poincaré algebra in the $c \to \infty$ limit. However, we will be concerned with massive theories which are not faithfully represented in Eq. (3.9). We will thus be interested in defining a Casimir invariant for this group and will deal with projective representations of the Galilean group. This is equivalent to the unitary representations of the non-trivial central extension of the universal Galilean group by a one dimensional Lie group. The corresponding Lie group is known in the literature as the Bargmann group [33]. In contrast to the Galilean group, this group has a central charge M which is a Casimir invariant i.e. commutes with all other generators of the group Eq. (3.8). The last commutation relation of Eq. (3.9) will be modified for the Bargmann group as follows,

$$[K_a, P_b] = M\delta_{ab} \tag{3.10}$$

In the next section we will discuss our proposed formalism to attain NRDI.

3.3 Gauging the Galilean symmetry

In [9] the Galilean symmetry was localized for a non-relativistic field theoretic model. In this section, the localization procedure and its key results will be discussed in detail. The first step of the procedure involves the consideration of a general non-relativistic action invariant under global Galilean transformations,

$$S = \int dt d^3x \mathcal{L}\left(\phi, \partial_t\phi, \partial_k\phi\right) \tag{3.11}$$

where the index 't' and '$k = 1, 2, 3$' denote time and spatial coordinates respectively. In covariant notation, these can be represented collectively by μ. However, in the localization of the Galilean transformations it will be important to maintain the distinction between space and time. This will be further elaborated below.

The infinitesimal global Galilean transformations under which the action Eq. (3.11) is invariant, will be parametrized in the following way :

$$x^\mu \longrightarrow x^\mu + \xi^\mu \tag{3.12}$$

where

$$\xi^0 = -\epsilon, \qquad \xi^i = \epsilon^i + \omega^i{}_j x^j - v^i t = \eta^i - v^i t \tag{3.13}$$

ϵ, ϵ^i, ω^{ij} and v^i are the parameters corresponding to the time translation, space translation, spatial rotation and boost respectively. The rotation parameters ω^{ij} are antisymmetric under the interchange of indices. These parameters are constant for the global transformation. The invariance of the action Eq. (3.11) under Eq. (3.12) requires that $\Delta\mathcal{L}$ either vanish or be a total derivative (Eq. (2.33)). This will be ensured by two conditions, the first being that $\partial_\mu \xi^\mu = 0$ for the global case. The second condition is that the field and its derivatives will transform under Eq. (3.12) as follows,

$$\begin{aligned}
\delta_0^G \phi &= \epsilon\partial_t\phi - \eta^i\partial_i\phi + tv^i\partial_i\phi - imv^i x_i\phi \\
\delta_0^G \partial_k\phi &= \epsilon\partial_t(\partial_k\phi) - \left(\eta^i - v^i t\right)\partial_i(\partial_k\phi) - imv^i\partial_k(x_i\phi) + \omega_k{}^m\partial_m\phi \\
\delta_0^G \partial_t\phi &= \epsilon\partial_t(\partial_t\phi) - (\eta^i - tv^i)\partial_i(\partial_t\phi) - imv^i x_i\partial_t\phi + v^i\partial_i\phi
\end{aligned} \tag{3.14}$$

Note that under Galilean boosts the field transforms as,

$$\phi'(t', x') = e^{\frac{i}{2}mv^2 t - imv_i x^i}\phi(t, x) \tag{3.15}$$

As our procedure deals with infinitesimal transformations, we will be interested in the term linearly proportional to the boost parameter. When we localize the Galilean transformations, the transformation parameters ϵ, ϵ^i, ω^{ij} and v^i are no longer constants, and hence $\partial_\mu\xi^\mu \neq 0$. Keeping in mind the nature of non-relativistic spacetime, the most general local transformations are given by

$$t \to t - \epsilon(t), \qquad x^i \to x^i + \epsilon^i(x,t) + \omega^i{}_j(x,t)x^j - v^i(x,t)t \tag{3.16}$$

The action which was invariant under global Galilean transformations is clearly no longer invariant under the local ones. This follows from $\partial_\mu\xi^\mu \neq 0$ and hence the derivatives of ϕ do not vary as stated in Eq. (3.14).

$$\begin{aligned}
\delta_0\partial_k\phi &= -\xi^\mu\partial_\mu(\partial_k\phi) - \partial_k\xi^\mu\partial_\mu\phi - im\partial_k(v^i x_i\phi) \\
\delta_0\partial_t\phi &= -\xi^\mu\partial_\mu(\partial_t\phi) - \partial_t\xi^\mu\partial_\mu\phi - imx_i\partial_t(v^i\phi)
\end{aligned} \tag{3.17}$$

To retain the invariance, the next step involves the introduction of additional gauge fields which are defined through covariant derivatives. Like other gauge theories, the gauge covariant derivatives with respect to the global coordinates are defined as,

$$D_k\phi = \partial_k\phi + iB_k\phi$$
$$D_t\phi = \partial_t\phi + iB_t\phi \tag{3.18}$$

where B_k and B_t are new fields. Using Eq. (3.17) and Eq. (3.18) the variations of these covariant derivatives can be derived as follows,

$$\begin{aligned}\delta_0 D_t\phi =& -\xi^\mu\partial_\mu D_t\phi - \partial_t\xi^\mu D_\mu\phi - imv^i x_i D_t\phi\\ & i\phi(\delta_0 B_t + \xi^\mu\partial_\mu B_t + \partial_t\xi^\mu B_\mu - m\dot{v}^i x_i)\\ \delta_0 D_k\phi =& -\xi^\mu\partial_\mu D_k\phi - \partial_k\xi^\mu D_\mu\phi - imv^i x_i D_k\phi\\ & i\phi(\delta_0 B_k + \xi^\mu\partial_\mu B_k + \partial_k\xi^\mu B_\mu - mv_k - m\partial_k v^i x_i)\end{aligned} \tag{3.19}$$

It can be noted that choosing $\delta_0 B_t$, $\delta_0 B_k$ appropriately to make the term in the parenthesis of Eq. (3.19) vanish is not sufficient to restore invariance as $D_k\phi$ and $D_t\phi$ do not vary like Eq. (3.14). This is similar to what one observes in PGT. In order to remedy this, we proceed in two steps inspired by PGT. First, local spatial coordinates 'x^a' (a =1,2,3) are introduced, which will also help in providing a geometrical framework to the local Galilean transformations. Local spatial coordinates $x^a, a = 1, 2$ are trivially connected with the global coordinates x^i by,

$$x^a = \delta^a_i x^i \tag{3.20}$$

We then introduce additional gauge fields by defining the local covariant derivatives in the following way

$$\tilde{D}_0\phi = {\Sigma_0}^0 D_t\phi + {\Sigma_0}^k D_k\phi$$
$$\tilde{D}_a\phi = {\Sigma_a}^k D_k\phi \tag{3.21}$$

$\tilde{D}_a\phi$ now transforms as required,

$$\delta_0\tilde{D}_a\phi = -\xi^\mu\partial_\mu\tilde{D}_a\phi - imv^b x_b\tilde{D}_a\phi - imv_a\phi + {\omega_a}^b\tilde{D}_b\phi \tag{3.22}$$

if the fields B_k and ${\Sigma_a}^k$ vary according to,

$$\begin{aligned}\delta_0^G B_k &= -\xi^\mu\partial_\mu B_k - \partial_k\xi^i B_i + m\partial_k v^i x_i + m\left(v_k - {\Lambda_k}^a v_a\right)\\ \delta_0^G {\Sigma_a}^k &= -\xi^\mu\partial_\mu{\Sigma_a}^k + {\Sigma_a}^i\partial_i\xi^k + {\omega_a}^b{\Sigma_b}^k\end{aligned} \tag{3.23}$$

Similarly $\tilde{D}_0\phi$ would do the same,

$$\delta_0\tilde{D}_0\phi = -\xi^\mu\partial_\mu\tilde{D}_0\phi - imv^b x_b\tilde{D}_0\phi + v^b\tilde{D}_b\phi \tag{3.24}$$

provided the variations of $B_t, {\Sigma_0}^0$ and ${\Sigma_0}^k$ satisfy,

$$\begin{aligned}\delta_0^G B_t &= -\xi^\mu\partial_\mu B_t - \partial_t\xi^\mu B_\mu + m{\Sigma_0}^k{\Lambda_k}^a v_a + m\dot{v}^i x_i\\ \delta_0^G{\Sigma_0}^0 &= -\xi^0\partial_t{\Sigma_0}^0 + \partial_t\xi^0{\Sigma_0}^0\\ \delta_0^G{\Sigma_0}^k &= -\xi^\mu\partial_\mu{\Sigma_0}^k + \partial_i\xi^k{\Sigma_0}^i + \partial_t\xi^k + \frac{1}{{\Sigma_0}^0}v^b{\Sigma_b}^k\end{aligned} \tag{3.25}$$

Thus it can be observed that the local covariant derivatives transform as required. In the following step, the partial derivatives in the action Eq. (3.11) are replaced by these local covariant derivatives.

$$\mathcal{L}(\phi, \partial_t\phi, \partial_k\phi) \to \mathcal{L}'(\phi, \tilde{D}_0\phi, \tilde{D}_a\phi)$$

$\mathcal{L}'$ now satisfies,

$$\delta_0\mathcal{L}' + \xi^\mu\partial_\mu\mathcal{L}' = 0 \tag{3.26}$$

Note that the condition Eq. (2.33) still does not hold. As the factor $\partial_\mu\xi^\mu$ in Eq. (2.33) comes from the Jacobian of the coordinate transformations, the invariance of the total action under the local transformations can be accounted for by a change in the measure. If we rescale the Lagrangian density by Λ,

$$\mathcal{L}' \to \Lambda\mathcal{L}' \tag{3.27}$$

then to preserve the invariance under the local Galilean transformation, Λ has to satisfy,

$$\delta_0\Lambda + \xi^\mu\partial_\mu\Lambda + \partial_\mu\xi^\mu\Lambda = 0 \tag{3.28}$$

The relation Eq. (3.28) follows from Eq. (2.33). It can be demonstrated that the appropriate Jacobian for the Galilean transformations is,

$$\Lambda = \frac{\det {\Lambda_k}^a}{{\Sigma_0}^0} \tag{3.29}$$

where ${\Lambda_k}^a$ is the inverse of ${\Sigma_a}^k$, satisfying the relations

$${\Lambda_k}^a{\Sigma_a}^l = \delta_k^l \ ; \ {\Sigma_a}^k{\Lambda_k}^b = \delta_a^b \tag{3.30}$$

The variation of ${\Lambda_k}^a$ can be derived using Eq. (3.23),

$$\delta_0{\Lambda^a}_k = -\xi^\mu\partial_\mu\Lambda^a_k - {\Lambda^a}_l\partial_k\xi^l - {\omega_b}^a{\Lambda^b}_l \tag{3.31}$$

Using Eq. (3.25) and the following relation,

$$\delta_0 M = -M{\Lambda^a}_k\ \delta_0{\Sigma_a}^k, \quad M = \det {\Lambda_k}^a \tag{3.32}$$

One can see that the Λ Eq. (3.29) satisfies Eq. (3.28).

The result of our whole procedure is the following action which is invariant under local Galilean transformations,

$$S = \int dt d^3x\ \Lambda\ \mathcal{L}\left(\phi, \tilde{D}_0\phi, \tilde{D}_a\phi\right) \tag{3.33}$$

As a concrete example we will consider the complex Schrödinger scalar field on Euclidean space,

$$S = \int dt \int d^3x \left[\frac{i}{2}\left(\phi^*\partial_t\phi - \phi\partial_t\phi^*\right) - \frac{1}{2m}\partial_k\phi^*\partial_k\phi\right] \tag{3.34}$$

This action is invariant under the global Galilean transformations Eq. (3.12) and the U(1) gauge transformation considered in section Section 3.1. Following the localization procedure described above, the Schrödinger action invariant under the local Galilean transformations is,

$$S=\int dt\int d^3x\ \Lambda\ \left[\frac{i}{2}\left(\phi^*\tilde{D}_0\phi-\phi\tilde{D}_0\phi^*\right)-\frac{1}{2m}\tilde{D}_a\phi^*\tilde{D}^a\phi\right] \tag{3.35}$$

In the following section we will demonstrate the NRDI of this action.

3.4 Application of the localisation procedure

The importance of the construction Eq. (3.35) will be appreciated in this section. This construction naturally leads to a 3-d spatial diffeomorphism invariant action. To demonstrate that Eq. (3.35) corresponds to a matter field coupled to a curved background, we need the spatial metric to be manifest in the action. It is instructive to recall a property of differential manifolds equipped with a metric, that the determinant of the metric tensor is equivalent to the square of the Jacobian. This property is reflected in the invariant measure in the relativistic context, which is given by $\sqrt{|h|}d^nx$, where $|h|$ is the (positive) determinant of the metric. In Eq. (3.35) the invariant measure is given by Eq. (3.29), which suggests that the 'Σ' and 'Λ' fields are related with the metric. In the following subsection, it will be demonstrated that this is indeed the case.

3.4.1 Non-relativistic spatial diffeomorphism invariance

In order to achieve 3-d spatial diffeomorphism we have to consider constant time indicating that time translation $\xi^0=-\epsilon$ should be vanishing. Then the local Galilean transformation is equivalent to the transformation,

$$x^i\longrightarrow x^i+\xi^i\left(\mathbf{x},\mathbf{t}\right) \tag{3.36}$$

where ξ^i is an arbitrary function of $\mathbf{x}$ and t. From the set of transformations Eq. (3.25) one can find that when $\epsilon=0,\Sigma_0{}^0=$ constant. Without any loss of generality we can take $\Sigma_0{}^0=1$. Note that even with this choice of time, Eq. (3.36) actually represents time dependent diffeomorphisms.

To get a geometrical picture we introduce the spatial 'metric tensor' h_{ij},

$$h_{ij}=\delta_{cd}\Lambda^c{}_i\Lambda^d{}_j \tag{3.37}$$

Using the transformation relation of $\Lambda^a{}_k$ from Eq. (3.31) the metric transforms as,

$$\delta_0 h_{ij}=-\xi^k\partial_k h_{ij}-h_{ik}\partial_j\xi^k-h_{kj}\partial_i\xi^k \tag{3.38}$$

We find that the transformation of h_{ij} is the same as those under a diffeomorphism. Thus $\Lambda^a{}_k$ can be identified with the inverse vierbein. The change in measure Λ will be $\sqrt{h}$.

Using these in Eq. (3.33) the local Galilean invariant action reduces to,

$$S = \int dtd^3x\sqrt{h}\mathcal{L}\left(\phi, \tilde{D}_t\phi, \tilde{D}_a\phi\right) \tag{3.39}$$

which is invariant under 3-d non-relativistic spatial diffeomorphism Eq. (3.36). We can appreciate the action Eq. (3.39) as matter fields coupled to a curved background. As an example, the action of the complex Schrödinger field on a curved background will be,

$$S = \int dt \int d^3x\sqrt{h}\left[\frac{i}{2}\left(\phi^*\tilde{D}_t\phi - \phi\tilde{D}_t\phi^*\right) - \frac{1}{2m}\tilde{D}_a\phi^*\tilde{D}_a\phi\right] \tag{3.40}$$

Now one can simplify the last term in the parenthesis as,

$$\tilde{D}_a\phi^*\tilde{D}_a\phi = \delta^{ab}\Sigma_a{}^k\Sigma_b{}^l D_k\phi^* D_l\phi = h^{kl}D_k\phi^* D_l\phi \tag{3.41}$$

where

$$h^{kl} = \delta^{ab}\Sigma_a{}^k\Sigma_b{}^l \tag{3.42}$$

is the inverse metric satisfying,

$$h^{kl}h_{ln} = \delta^k_n \tag{3.43}$$

The $\Sigma_a{}^k$ are the vierbeins connecting the tangent space and the curved 3-d space, on which the theory is now formulated. Using Eq. (3.41) in Eq. (3.40) we obtain the most general 3-d diffeomorphism invariant Schrödinger action as,

$$S = \int dt \int d^3x\sqrt{h}\left[\frac{i}{2}\left(\phi^*\tilde{D}_t\phi - \phi\tilde{D}_t\phi^*\right) - \frac{1}{2m}g^{kl}D_k\phi^* D_l\phi\right] \tag{3.44}$$

Note that it is very easy to take the flat limit of Eq. (3.44); one simply replaces g^{kl} by δ^{kl} and substitutes covariant derivatives by ordinary derivatives. This immediately reproduces Eq. (3.34). We will provide the total covariant expression for Eq. (3.44) in Chapter 5, where the geometric properties of the background will be identified with the NC geometry.

In the following chapter we will extend this localization procedure to consider vector fields in addition to the scalar fields. It will be demonstrated that spatial diffeomorphism will still be preserved.

Chapter 4

Localisation of Galilean symmetry for vector fields

In the previous chapter we explicitly discussed the localisation of spacetime symmetries, specifically the Galilean symmetry, for non-relativistic scalar fields. We have successfully derived a 3-d spatial diffeomorphism invariant theory. In this chapter we will include gauge fields in our formalism. As an example first we consider the scalar field interacting with an external gauge field in $2+1$ dimensions. Following this we will explore spatial diffeomorphism in dynamical gauge theories - in particular the Chern-Simons theory.

4.1 Gauging the Galilean symmetry of a model with scalar and vector fields

We start with a theory where the set of fields contain a gauge field corresponding to electromagnetic interaction in addition to the scalar field. In other words, we consider the non-relativistic complex scalar fields minimally interacting with a vector gauge field in $2+1$ dimensions, invariant under global Galilean transformations Eq. (3.12). The action will be of the following form,

$$S = \int dt\ d^2x\ \mathcal{L}\left(\phi, \partial_\mu\phi, A_\mu, \partial_\mu A_\nu\right) \tag{4.1}$$

The action Eq. (4.1) is assumed to be invariant under the local Abelian gauge transformations,

$$\begin{aligned} \phi &\to \phi + i\alpha\phi \\ A_\mu &\to A_\mu - \partial_\mu\alpha \end{aligned} \tag{4.2}$$

Under the global Galilean transformation Eq. (3.12) the action Eq. (4.1) would be invariant provided the scalar field transforms as Eq. (3.14) and the gauge field transforms as [50],

$$\begin{aligned}
\delta_0 A_0 &= \epsilon\partial_0 A_0 - \eta^l\partial_l A_0 + tv^l\partial_l A_0 + v^l A_l = -\xi^\mu\partial_\mu A_0 + v^l A_l \\
\delta_0 A_i &= \epsilon\partial_0 A_i - \eta^l\partial_l A_i + tv^l\partial_l A_i + \omega_i{}^l A_l = -\xi^\mu\partial_\mu A_i + \omega_i{}^l A_l
\end{aligned} \tag{4.3}$$

where $\eta^i = \epsilon^i + \omega^i{}_j x^j$. A_k transforms as a vector under rotation while A_0 transforms as a scalar under the same. Consequently, the derivatives acting on A_0 and A_i will vary as,

$$\begin{aligned}
\delta_0\partial_k A_0 &= -\xi^\mu\partial_\mu(\partial_k A_0) + \omega_k{}^l\partial_l A_0 + v^l\partial_k A_l \\
\delta_0\partial_0 A_0 &= -\xi^\mu\partial_\mu(\partial_0 A_0) + v^l\partial_l A_0 + v^l\partial_0 A_l
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\delta_0\partial_k A_i &= -\xi^\mu\partial_\mu(\partial_k A_i) + \omega_k{}^l\partial_l A_i + \omega_i{}^l\partial_k A_l \\
\delta_0\partial_0 A_k &= -\xi^\mu\partial_\mu(\partial_0 A_k) + v^l\partial_l A_k + \omega_k{}^l\partial_0 A_l
\end{aligned} \tag{4.5}$$

These transformations ensure the following relation,

$$\delta_0\mathcal{L} + \xi^\mu\partial_\mu\mathcal{L} = 0 \tag{4.6}$$

For global transformations $\partial_\mu\xi^\mu = 0$. Together they keep $\delta S = 0$ under the global Galilean transformations, where S is given by Eq. (4.1).

Now we make the transformations local Eq. (3.16). Similar to the case of the scalar field when the parameters of the transformations are local, the partial derivatives of ϕ, A_0, A_i with respect to space and time will no longer transform as Eq. (3.17), Eq. (4.4), Eq. (4.5). Following the localization procedure stated in the previous chapter, one needs to introduce covariant derivatives which will transform covariantly as Eq. (3.17), Eq. (4.4), Eq. (4.5) with respect to the local coordinates. The first step in the process of localization is to introduce covariant derivatives with respect to the global coordinates. We have already introduced the gauge fields B_μ in Section 3.3 to define covariant derivatives Eq. (3.18) acting on the complex scalar field ϕ with respect to global coordinate.

In addition, new gauge fields C_μ, F_μ will be introduced here to define the global covariant derivatives for the fields A_μ as,

$$\begin{aligned}
\tilde{D}_\mu A_0 &= \partial_\mu A_0 + iC_\mu A_0 \\
\tilde{D}_\mu A_i &= \partial_\mu A_i + iF_\mu A_i
\end{aligned} \tag{4.7}$$

Note that different sets of gauge fields are introduced for A_0 and A_i due to the nature of Galilean spacetime. As these global covariant derivatives do not ensure the invariance of the action under local transformations, in the next step we define the local coordinates and then the covariant derivatives with respect to them. For the complex scalar field these local covariant derivatives were already defined in Eq. (3.21). Introducing additional fields $\Sigma_0{}^0(t), \Sigma_0{}^k(x^0, \mathbf{r}), \Sigma_a{}^k(x^0, \mathbf{r})$

in the process we found that the local covariant derivative transform covariantly Eq. (3.22), Eq. (3.24).

We will follow a similar procedure to construct the appropriate local covariant derivatives for the gauge fields A_μ,

$$\begin{aligned}
\tilde{D}_a A_{\bar{0}} &= \Sigma_a{}^k D_k A_0 \\
\tilde{D}_{\bar{0}} A_{\bar{0}} &= \Sigma_0{}^0 D_0 A_0 + \Sigma_0{}^k D_k A_0 \\
\tilde{D}_a A_b &= (\Sigma_a{}^k D_k A_i)\delta^i{}_b \\
\tilde{D}_{\bar{0}} A_b &= (\Sigma_0{}^0 D_0 A_i + \Sigma_0{}^k D_k A_i)\delta^i{}_b
\end{aligned} \tag{4.8}$$

where we have denotes the local time coordinate by $\bar{0}$. Plugging the expression of $\delta_0\Sigma_a{}^k, \delta_0\Sigma_0{}^k, \delta_0\Sigma_0{}^0$ from Eq. (3.23), Eq. (3.25) in the variation of the local covariant derivatives one can observe that these will transform as required,

$$\begin{aligned}
\delta_0(\tilde{D}_a A_{\bar{0}}) &= -\xi^\mu \partial_\mu (\nabla_a A_{\bar{0}}) + \omega_a{}^b \nabla_b A_{\bar{0}} + v^b \nabla_a A_b \\
\delta_0(\tilde{D}_{\bar{0}} A_{\bar{0}}) &= -\xi^\mu \partial_\mu (\nabla_{\bar{0}} A_{\bar{0}}) + v^b \nabla_b A_{\bar{0}} + v^b \nabla_{\bar{0}} A_b \\
\delta_0(\tilde{D}_a A_b) &= -\xi^\mu \partial_\mu \partial_0 (\nabla_a A_b) + \omega_a{}^c \nabla_c A_b + \omega_b{}^c \nabla_a A_c \\
\delta_0(\tilde{D}_{\bar{0}} A_b) &= -\xi^\mu \partial_\mu \partial_0 (\nabla_{\bar{0}} A_b) + v^a \nabla_a A_b + \omega_b{}^c \nabla_{\bar{0}} A_c
\end{aligned} \tag{4.9}$$

provided

$$\begin{aligned}
\delta_0 C_0 &= -\xi^\mu \partial_\mu C_0 - \partial_t \xi^\mu C_\mu + i A_0{}^{-1} \dot{v}^l A_l \\
\delta_0 C_k &= -\xi^\mu \partial_\mu C_k - \partial_k \xi^i C_i + i A_0{}^{-1} \partial_k (v^l) A_l \\
\delta_0 F_0 &= -\xi^\mu \partial_\mu F_0 - \partial_t \xi^\mu F_\mu \\
\delta_0 F_k &= -\xi^\mu \partial_\mu F_k - \partial_k \xi^l F_l
\end{aligned} \tag{4.10}$$

Certain interesting features can be noticed in the variation of new fields that define the covariant derivatives acting on the gauge field. The variation of the C fields have an extra boost parameter dependent term in contrast to the variation of the F fields.

We can now replace the partial derivatives in the action with the local covariant derivatives. However $\partial_\mu \xi^\mu \neq 0$ under local Galilean transformations and as before, we need to correct for the measure Eq. (3.29). Thus the action invariant under the local Galilean transformations will be,

$$S = \int dt \; d^2x \; \Lambda \; \mathcal{L}\left(\phi, \tilde{D}_\alpha \phi, A_\alpha, \tilde{D}_\alpha A_\beta\right) \qquad (\alpha, \beta \equiv \bar{0}, a) \tag{4.11}$$

In the following subsection this action Eq. (4.11) will be reinterpreted as one on a curved background. This will naturally lead to a diffeomorphism invariant theory in space similar to the case of the scalar field Section 3.4.1.

4.2 Emergence of spatial diffeomorphism

We will now demonstrate that our formalism leads to a diffeomorphism invariant theory of scalar and gauge fields in 2-d space. We thus consider that the time translation in Eq. (3.16) vanishes, which implies ${\Sigma_0}^0 =$ constant. This allows us the freedom to fix ${\Sigma_0}^0 = 1$. The local Galilean transformations are then equivalent to the general coordinate transformations in space Eq. (3.36). This indicates the possibility of reinterpreting the invariance of Eq. (4.11) under Eq. (3.16) as diffeomorphism invariance in curved space. The resulting theory Eq. (4.11) in the previous section was formulated in terms of locally flat coordinates. When the background space is curved, the local flat space is just the tangent space at the given point of spacetime. In this new interpretation the coordinates labelled by '$a, b, c \cdots$' define an orthogonal basis for the tangent space, while those labelled by '$i, j, k, \cdots$' define the coordinate basis for the curved space. In Cartan's formalism, the connection between the two is established through the vierbeins. The fields ${\Sigma_a}^k$ can be reinterpreted as the vierbeins, as already observed in Section 3.4.1.

We will now reconsider the transformation of ${\Sigma_a}^k$ obtained from Eq. (3.23) under the assumption $\epsilon = 0$,

$$\delta_0 {\Sigma_a}^k = {\Sigma_a}^i \partial_i \xi^k - \xi^i \partial_i {\Sigma_a}^k + {\omega_a}^b {\Sigma_b}^k \tag{4.12}$$

One can notice the dual aspects of the transformation. With respect to the coordinates x^i it satisfies the transformation rules of a contravariant vector under the general coordinate transformations whereas with respect to the coordinates x^a it is a local rotation. In a similar manner, we can observe from the variation of the inverse vierbein ${\Lambda_k}^a$ Eq. (3.31) that corresponding to its lower index k it transforms as covariant vector under diffeomorphisms, while corresponding to its local index a it transforms as an euclidean vector under rotation. It will thus be reasonable to propose the following connection between local and global coordinates in the overlapping patch,

$$dx_a = {\Sigma_a}^k dx_k \tag{4.13}$$

Note that contrary to Eq. (3.20), the above connection has become non-trivial due to the geometric interpretation.

Now we will follow Section 3.4.1 step by step. Thus we will construct a metric (and its inverse) for the 2-d spatial manifold from the fields ${\Sigma_a}^k$ and its inverse ${\Lambda_k}^a$,

$$h_{ij} = \delta_{cd} {\Lambda_i}^c {\Lambda_j}^d, \quad h^{kl} = \delta^{ab} {\Sigma_a}^k {\Sigma_b}^l \tag{4.14}$$

The above developments will modify the action Eq. (4.11) as follows,

$$S = \int dt d^2x \sqrt{h} \mathcal{L} \left(\phi, \tilde{D}_\alpha \phi, A_\alpha, \tilde{D}_\alpha A_\beta \right) \tag{4.15}$$

Rewriting the fields of Eq. (4.15) in global coordinates we get a covariant description of scalar fields interacting with the gauge field on curved backgrounds.

Thus our theory gives a structural algorithm of constructing spatially diffeomorphism invariant theories from globally Galilean invariant theories with the general structure of Eq. (4.1). To establish this analogy we have to see how the transformations of the fields and the covariant derivatives obtained from the localization procedure in the previous section can be reinterpreted in the backdrop of curved space. Though we are working with vanishing time translations, the appearance of time in the diffeomorphism parameter ξ makes the time arrow relative at different points of curved space. Unlike scalar fields the time component of the vectors in the local coordinates will not be simply equal with that of the curved space. Using equations Eq. (3.14) and Eq. (4.3) we can write the variations of ϕ, $A_{\bar{0}}$ and A_a in the local coordinates as,

$$\begin{aligned}
\delta_0 \phi &= -\xi^a \partial_a \phi - i m v^a x_a \phi \\
\delta_0 A_{\bar{0}} &= -\xi^b \partial_b A_{\bar{0}} + v^b A_b \\
\delta_0 A_a &= -\xi^b \partial_b A_a + \omega_a{}^b A_b
\end{aligned} \tag{4.16}$$

In terms of these we will define the appropriate transformations in the curved space. In this context, the mapping can only be achieved in the overlap of the two systems i.e in the neighbourhood of origin of the local system.

We begin with the scalar field ϕ whose transformation in curved space is given by,

$$\delta_0 \phi = -\xi^i \partial_i \phi \tag{4.17}$$

This follows from requiring that the two descriptions match in the neighbourhood of the origin of the local coordinate system. Hence the last term of the corresponding equation of Eq. (4.16) does not appear in Eq. (4.17). Spatial components of the vector field $\mathbf{A}$ in local and curved space are connected by a relation similar to Eq. (4.13),

$$A_a = \Sigma_a{}^k A_k \tag{4.18}$$

Using the variation of A_a from Eq. (4.16) and that of $\Sigma_a{}^k$ from Eq. (3.23) we can get the transformation of A_k in the curved basis,

$$\delta_0 A_k = -\xi^i \partial_i A_k - \partial_k \xi^i A_i \tag{4.19}$$

It is evident from Eq. (4.19) that A_k transforms like a covariant vector on a curved background. In deriving Eq. (4.19) we have used the following operator relation,

$$\begin{aligned}
\xi_a \frac{\partial}{\partial_x^a} &= \xi_a \frac{\partial x_i}{\partial x_a} \frac{\partial}{\partial x_i} \\
&= \Sigma_a{}^k \xi_k \Lambda_i{}^a \frac{\partial}{\partial x_i} \\
&= \xi_i \frac{\partial}{\partial_i}
\end{aligned} \tag{4.20}$$

which has been established using Eq. (4.13).

It has already been emphasized that despite our choice of time the spatial diffeomorphism parameters are time dependent. This is particularly noted through the time component of the vector field, which is related to the time component in curved space through the following relation,

$$A_{\bar{0}} = \Sigma_0{}^{\mu} A_{\mu} = A_0 + \Sigma_0{}^{k} A_k \tag{4.21}$$

The transformation rule for A_0 can similarly be worked out using the variations of $A_{\bar{0}}$, $\Sigma_0{}^k$ and A_k from Eq. (4.16), Eq. (3.25), Eq. (4.19),

$$\delta_0 A_0 = -\xi^i \partial_i A_0 - \dot{\xi}^i A_i \tag{4.22}$$

Given these transformations for the basic fields, we now need to define the appropriate covariant derivatives with respect to the curved coordinates corresponding to the local covariant derivatives $\tilde{D}_{\bar{0}}\phi$, $\tilde{D}_a\phi$, $\tilde{D}_a A_b$, $\tilde{D}_{\bar{0}} A_a$, $\tilde{D}_a A_{\bar{0}}$ and $\tilde{D}_{\bar{0}} A_{\bar{0}}$. We denote these respectively by $\mathcal{D}_0\phi$, $\mathcal{D}_k\phi$, $\mathcal{D}_k A_l$, $\mathcal{D}_0 A_l$, $\mathcal{D}_k A_0$ and $\mathcal{D}_0 A_0$. The following definitions are proposed:

$$\begin{aligned}
\tilde{D}_a\phi &= \Sigma_a{}^k \mathcal{D}_k\phi \\
\tilde{D}_{\bar{0}}\phi &= \mathcal{D}_0\phi + \Sigma_0{}^k \mathcal{D}_k\phi \\
\tilde{D}_a A_b &= \Sigma_a{}^k \Sigma_b{}^l \mathcal{D}_k A_l \\
\tilde{D}_{\bar{0}} A_a &= \Sigma_a{}^k \left(\mathcal{D}_0 A_k + \Sigma_0{}^l \mathcal{D}_l A_k\right) \\
\tilde{D}_a A_{\bar{0}} &= \Sigma_a{}^k \left(\mathcal{D}_k A_0 + \Sigma_0{}^l \mathcal{D}_k A_l\right) \\
\tilde{D}_{\bar{0}} A_{\bar{0}} &= \mathcal{D}_0 A_0 + \Sigma_0{}^k \mathcal{D}_k A_0 + \Sigma_0{}^k \mathcal{D}_0 A_k + \Sigma_0{}^k \Sigma_0{}^l \mathcal{D}_k A_l
\end{aligned} \tag{4.23}$$

The transformation laws of these derivatives can be obtained from the transformations rules provided in Eq. (3.22), Eq. (3.24) and Eq. (4.9). Here we will explicitly calculate the transformation of $\mathcal{D}_k\phi$. Taking the form variation of both sides of the first equation of Eq. (4.23) we get,

$$\delta_0\left(\tilde{D}_a\phi\right) = \left(\delta_0 \Sigma_a{}^k\right) \mathcal{D}_k\phi + \Sigma_a{}^k \left(\delta_0 \mathcal{D}_k\phi\right) \tag{4.24}$$

From (Eq. (3.22)) we write

$$\delta_0\left(\tilde{D}_a\phi\right) = -\xi^b \partial_b \left(\nabla_a\phi\right) - imv^b \nabla_a \left(x_b\phi\right) + \omega_a{}^b \nabla_b\phi \tag{4.25}$$

Substituting this result in Eq. (4.24) and using the transformation of $\Sigma_a{}^k$, we get the transformation $\delta_0 \mathcal{D}_k\phi$. Working analogously with the other covariant derivatives of Eq. (4.23), we find the following transformation rules,

$$\begin{aligned}
\delta_0 \mathcal{D}_k\phi &= -\xi^i \partial_i \left(\mathcal{D}_k\phi\right) - \partial_k \xi^i \mathcal{D}_i\phi \\
\delta_0 \mathcal{D}_0\phi &= -\xi^i \partial_i \left(\mathcal{D}_0\phi\right) - \dot{\xi}^k \mathcal{D}_k\phi \\
\delta_0 \mathcal{D}_k A_l &= -\xi^i \partial_i \left(\mathcal{D}_k A_l\right) - \partial_k \xi^m \mathcal{D}_m A_l - \partial_l \xi^m \mathcal{D}_k A_m \\
\delta_0 \mathcal{D}_0 A_k &= -\xi^i \partial_i \left(\mathcal{D}_0 A_k\right) - \partial_k \xi^l \mathcal{D}_0 A_l - \dot{\xi}^l \mathcal{D}_l A_k \\
\delta_0 \mathcal{D}_k A_0 &= -\xi^i \partial_i \left(\mathcal{D}_k A_0\right) - \partial_k \xi^l \mathcal{D}_l A_0 - \dot{\xi}^l \mathcal{D}_k A_l \\
\delta_0 \mathcal{D}_0 A_0 &= -\xi^i \partial_i \left(\mathcal{D}_0 A_0\right) - \dot{\xi}^k \left(\mathcal{D}_k A_0 + \mathcal{D}_0 A_k\right)
\end{aligned} \tag{4.26}$$

Eq. (4.26) demonstrates that the definitions of Eq. (4.23) transform canonically. We can thus formulate an action invariant under general coordinate transformations by the substitution of Eq. (4.15) and by replacing the covariant derivatives in the action Eq. (4.11) by the covariant derivatives of Eq. (4.23).

For explicit calculations we will require expressions for the derivatives $\mathcal{D}_k\phi, \mathcal{D}_0\phi$, $\mathcal{D}_kA_l, \mathcal{D}_0A_k, \mathcal{D}_kA_0$ in terms of the basic fields with well defined transformations. These expressions are obtained by requiring consistency with Eq. (4.26). Following this, we define the derivatives $\mathcal{D}_0\phi$ and $\mathcal{D}_k\phi$ as,

$$\begin{aligned}\mathcal{D}_0\phi &= \partial_0\phi + i\mathcal{B}_0\phi \\ \mathcal{D}_k\phi &= \partial_k\phi + i\mathcal{B}_k\phi\end{aligned} \tag{4.27}$$

where the transformation rules for the fields $\mathcal{B}_0$ and $\mathcal{B}_k$ are given by,

$$\begin{aligned}\delta_0\mathcal{B}_0 &= -\xi^i\partial_i\mathcal{B}_0 - \dot{\xi}^i\mathcal{B}_i \\ \delta_0\mathcal{B}_k &= -\xi^i\partial_i\mathcal{B}_k - \partial_k\xi^i\mathcal{B}_i\end{aligned} \tag{4.28}$$

We observe that $\mathcal{B}_k$ and $\mathcal{B}_0$ transform as the appropriate components of a covariant vector. The new vector fields $\mathcal{B}$ emerge from the localization prescription that leads to our formulation in curved space. We similarly define the action of these derivatives on the 'A's in the following way,

$$\begin{aligned}\mathcal{D}_iA_k &= \partial_iA_k + i\mathcal{B}_iA_k \\ \mathcal{D}_0A_k &= \partial_0A_k + i\mathcal{B}_0A_k \\ \mathcal{D}_kA_0 &= \partial_kA_0 + i\mathcal{B}_kA_0\end{aligned} \tag{4.29}$$

These can be seen to satisfy the transformation rules Eq. (4.26). We have thus successfully provided a detailed description of the fields and the covariant derivatives on the curved background. In the following section we will discuss a couple of applications of our general formalism. The first model we consider is that of a complex Schrödinger field theory in the presence of an external vector field. In the second model, we consider the effect of including a Chern-Simons term.

In addition to local Galilean invariance, one should also analyse the behaviour of the action Eq. (4.15) under gauge transformations. Globally, the combination $(\partial_\mu\phi + iA_\mu\phi)$ transforms under the gauge transformation as follows,

$$\partial_\mu\phi + iA_\mu\phi \to (1 + i\alpha)(\partial_\mu\phi + iA_\mu\phi) \tag{4.30}$$

When the Galilean symmetry is localized the partial derivatives $\partial_\mu\phi$ are replaced by $\tilde{D}_\alpha\phi$ $(\alpha = \bar{0}, a)$. Now the combination $(\tilde{D}_\alpha\phi + iA_\alpha\phi)$ transforms as the global one,

$$\tilde{D}_a\phi + iA_a\phi \to (1 + i\alpha)(\tilde{D}_a\phi + iA_a\phi) \tag{4.31}$$

provided we have the following gauge transformations of the basic fields ,

$$\phi \to \phi + i\alpha\phi, \quad A_a \to A_a - \tilde{D}_a\alpha, \quad A_{\bar{0}} \to A_{\bar{0}} - \tilde{D}_{\bar{0}}\alpha \tag{4.32}$$

where,

$$\tilde{D}_a\alpha = \Sigma_a{}^k\partial_k\alpha, \quad \tilde{D}_{\bar{0}}\alpha = \partial_0\alpha + \Sigma_0{}^m\partial_m\alpha \tag{4.33}$$

From Eq. (4.18), Eq. (4.21) and Eq. (4.33) we can analyse the behaviour of the external gauge field in curved space under the gauge transformation. It is given by

$$A_k \to A_k - \partial_k\alpha, \quad A_0 \to A_0 - \partial_0\alpha \tag{4.34}$$

and has the expected form suggested by Eq. (4.2).

4.3 Complex Schrödinger field in the presence of external vector field

An important application of spatial diffeomorphism is in the theory of the fractional quantum Hall effect [1–3]. Therefore we will start with the example of a non relativistic electron moving in an external gauge field given by the action,

$$S = \int dx^0 \int d^2x_k \left[\frac{i}{2}\left(\phi^*\Delta_0\phi - \phi\Delta_0\phi^*\right) - \frac{1}{2m}\Delta_k\phi^*\Delta_k\phi\right] \tag{4.35}$$

where

$$\begin{aligned} \Delta_0\phi &= \partial_0\phi + iA_0\phi \\ \Delta_k\phi &= \partial_k\phi + iA_k\phi \end{aligned} \tag{4.36}$$

and A_μ is the external gauge field. The theory Eq. (4.35) is invariant under global Galilean transformations Eq. (3.12), as can be checked explicitly. The theory Eq. (4.35) in addition is invariant under the local gauge transformations,

$$\phi \to \phi - i\Lambda\phi, \quad \phi^* \to \phi^* + i\Lambda\phi^*, \quad A_\mu \to A_\mu + \partial_\mu\Lambda \tag{4.37}$$

Simplifying Eq. (4.35) we can get,

$$\begin{aligned} S = \int dx^0 \int d^2x_k \bigg[&\frac{i}{2}\left(\phi^*\partial_0\phi - \phi\partial_0\phi^*\right) - \phi^*\phi A_0 - \frac{1}{2m}\partial_k\phi^*\partial_k\phi - \\ &\frac{{A_k}^2}{2m}\phi^*\phi + \frac{i}{2m}A_k(\phi^*\partial_k\phi - \phi\partial_k\phi^*)\bigg] \end{aligned} \tag{4.38}$$

According to our algorithm, replacing the partial derivatives by suitable local covariant derivatives and considering the change in measure, we obtain the corresponding theory invariant under local Galilean transformations Eq. (3.16),

$$\begin{aligned} S = \int dx^{\bar{0}} \int d^2x_a\Lambda \bigg[&\frac{i}{2}\left(\phi^*\tilde{D}_{\bar{0}}\phi - \phi\tilde{D}_{\bar{0}}\phi^*\right) - \frac{1}{2m}\tilde{D}_a\phi^*\tilde{D}_a\phi - \phi^*\phi A_{\bar{0}} \\ &-\frac{{A_a}^2}{2m}\phi^*\phi + \frac{i}{2m}A_a(\phi^*\tilde{D}_a\phi - \phi\tilde{D}_a\phi^*)\bigg] \end{aligned} \tag{4.39}$$

In the following we will consider spatial diffeomorphism ($\epsilon = 0$) where ${\Sigma_0}^0 = 1$. We can then transform our results in a geometric setting following the algorithm stated in the previous section.

Let us first consider the special case when the spatial diffeomorphism parameter ξ, is time independent. The third equation of Eq. (3.25) shows that, along with the time independence of ξ, ${\Sigma_0}^k = 0$ may be chosen. Under this condition, $\tilde{D}_{\bar{0}}\phi = \mathcal{D}_0\phi$ which follows from Eq. (4.23). Using this fact and other definitions from Eq. (4.23) the action Eq. (4.39) reduces to,

$$S = \int dx^0 \int d^2x\Lambda \left[\frac{i}{2}\left(\phi^*\mathcal{D}_0\phi - \phi\mathcal{D}_0\phi^*\right) - \phi^*\phi A_0 - {\Sigma_a}^k{\Sigma_a}^l\left(\frac{1}{2m}\mathcal{D}_k\phi^*\mathcal{D}_l\phi\right)\right. \\ \left. -{\Sigma_a}^k{\Sigma_a}^l\left(\frac{1}{2m}A_kA_l\phi^*\phi\right) + {\Sigma_a}^k{\Sigma_a}^l\left(\frac{i}{2m}A_k(\phi^*\mathcal{D}_l\phi - \phi\mathcal{D}_l\phi^*)\right)\right]$$

Using the definition of metric Eq. (4.14) this is reduced to a covariant theory in the curved space,

$$S = \int dx^0 d^2x\sqrt{h}\left[\frac{i}{2}\left(\phi^*(\mathcal{D}_0 + iA_0)\phi - \phi(\mathcal{D}_0 - iA_0)\phi^*\right)\right. \\ \left. -h^{kl}\frac{1}{2m}(\mathcal{D}_k - iA_k)\phi^*(\mathcal{D}_l + iA_l)\phi\right] \tag{4.40}$$

This action Eq. (4.40) can be rewritten as a non-relativistic diffeomorphism invariant action,

$$S = \int dx^0 d^2x\sqrt{h}\left[\frac{i}{2}\left(\phi^*\bar{D}_0\phi - \phi\bar{D}_0\phi^*\right) - h^{kl}\frac{1}{2m}\bar{D}_k\phi^*\bar{D}_l\phi\right] \tag{4.41}$$

by defining,

$$\begin{aligned} \bar{D}_0\phi &= \mathcal{D}_0\phi + iA_0\phi = \partial_0\phi + i\left(A_0 + \mathcal{B}_0\right)\phi \\ \bar{D}_k\phi &= \mathcal{D}_k\phi + iA_k\phi = \partial_k\phi + i\left(A_k + \mathcal{B}_k\right)\phi \end{aligned} \tag{4.42}$$

Note that under the local gauge transformation Eq. (4.37) the theory Eq. (4.41) is invariant provided the field $\mathcal{B}_\mu$ has the same gauge transformation as A_μ. It is reassuring to observe that under the restrictions assumed ($\epsilon = 0$ and ξ^i time independent), $\mathcal{B}_0$ transforms as A_0 and $\mathcal{B}_k$ as A_k which has been described earlier. We have thus observed that the result of localizing the Galilean symmetry of an interacting non-relativistic field theory in flat space also leads to an action invariant under general coordinate transformations in curved space. The model considered in this section is particularly important as it pertains to the effective action for the fractional quantum Hall effect. In particular, the transport properties of Hall systems can be affected on curved backgrounds (within the composite boson model) due to the presence of the spin connection and gravitational anomalies. The present formalism would allow us to carry over such investigations in the non-relativistic limit.

When the diffeomorphism parameter ξ^i is time dependent ${\Sigma_0}^k = 0$ is not admissible. Then the diffeomorphism invariant action in the curved space becomes,

$$S = \int dx^0 d^2x \sqrt{h} \left[\frac{i}{2} \left(\phi^* \bar{D}_0 \phi - \phi \bar{D}_0 \phi^* \right) - \frac{h^{kl}}{2m} \bar{D}_k \phi^* \bar{D}_l \phi \right. \\ \left. + \frac{i}{2} {\Sigma_0}^k \left(\phi^* \bar{D}_k \phi - \phi \bar{D}_k \phi^* \right) \right] \tag{4.43}$$

We can easily attain the flat limit by replacing the covariant derivatives by the ordinary derivatives and the spatial metric by δ_{ij}. A simple inspection of Eq. (4.43) and Eq. (4.35) confirms the above.

4.4 Inclusion of the Chern-Simons term in the action

Having considered the action of previous section, particularly in the context of Hall systems, it will be interesting to further involve the Chern-Simons (CS) term. Given the topological form of the CS action we expect it to be independent of any particular form of the metric (modulo boundary terms). However, some subtleties are involved in NRDI which we will now briefly elaborate on. The CS action is given by,

$$S_{CS} = \int d^3x \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \tag{4.44}$$

and can be coupled with both relativistic and non-relativistic models. It will be convenient to break the action into spatial and temporal parts,

$$S_{CS} = \int dt \int d^2x \ \frac{\kappa}{2} \epsilon^{ij} \left(A_0 \partial_i A_j - A_i \partial_0 A_j + A_i \partial_j A_0 \right) \tag{4.45}$$

It can be shown that Eq. (4.45) is invariant under the global Galilean transformation using the variations Eq. (4.3). Following the method to localize the Galilean transformation stated in previous section, we can get the corresponding action invariant under the the local Galilean transformations as,

$$S = \int dx^{\bar{0}} \int d^2x_a \Lambda \frac{\kappa}{2} \epsilon^{ab} \left(A_{\bar{0}} \tilde{D}_a A_b - A_a \tilde{D}_{\bar{0}} A_b + A_a \tilde{D}_b A_{\bar{0}} \right) \tag{4.46}$$

The algorithm given in Section 4.2 allows us to construct the diffeomorphism invariant action as follows,

$$S = \int dt d^2x \ \sqrt{h} \frac{\kappa}{2} \epsilon^{ab} {\Sigma_a}^k {\Sigma_b}^l \left[\left(A_0 \mathcal{D}_k A_l - A_k \mathcal{D}_0 A_l + A_k \mathcal{D}_l A_0 \right) \right. \\ \left. + {\Sigma_0}^m A_m \mathcal{D}_k A_l + {\Sigma_0}^m A_k \left(\mathcal{D}_l A_m - \mathcal{D}_m A_l \right) \right] \tag{4.47}$$

Note that ϵ^{ab} is a tensor under local transformations. Thus

$${\Sigma_a}^k {\Sigma_b}^l \epsilon^{ab} = \tilde{\epsilon}^{kl} \tag{4.48}$$

where $\tilde{\epsilon}^{kl}$ is the Levi Civita tensor in curved space. It is related to the tensor density ϵ^{kl} by,

$$\tilde{\epsilon}^{kl} = \frac{1}{\sqrt{h}} \epsilon^{kl} \tag{4.49}$$

Using the above equations the final form of the CS action in curved space is obtained as,

$$\begin{aligned} S = \int dt d^2x \ \frac{\kappa}{2} \epsilon^{kl} \left[(A_0 \mathcal{D}_k A_l - A_k \mathcal{D}_0 A_l + A_k \mathcal{D}_l A_0) \right. \\ \left. + \ \Sigma_0{}^m A_m \mathcal{D}_k A_l + \Sigma_0{}^m A_k \left(\mathcal{D}_l A_m - \mathcal{D}_m A_l \right) \right] \end{aligned} \tag{4.50}$$

We can simplify Eq. (4.50) substituting the derivatives $\mathcal{D}_\mu A_\nu$ from Eq. (4.29) and exploiting the antisymmetric property of ϵ^{kl},

$$\begin{aligned} S &= \int dt \ d^2x \frac{\kappa}{2} \ \epsilon^{kl} \left[2 \left(A_0 \partial_k A_l - A_k \partial_0 A_l + A_k \partial_l A_0 \right) \right. \\ &\qquad \left. + 2\Sigma_0{}^m [A_m \partial_k A_l + A_k (\partial_l A_m - \partial_m A_l)] \right] \\ &= \int dt \ d^2x \kappa [\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda] \end{aligned} \tag{4.51}$$

where we have defined the spacetime Levi-Civita density for the NC background as

$$\sqrt{h} \, \Sigma_0{}^\mu \Sigma_a{}^\nu \Sigma_b{}^\lambda \epsilon^{ab} = \epsilon^{\mu\nu\lambda} \tag{4.52}$$

We also note that the $\mathcal{B}$ field has dropped out from Eq. (4.51). Thus the topological invariance of the CS action is restored under non-relativistic diffeomorphisms, which has also allowed us to determine the form of the Levi-Civita tensor from the localization procedure.

Using Eq. (4.19), Eq. (4.22) and Eq. (4.26) it can be shown that the action Eq. (4.51) transforms under general coordinate transformations in the following way

$$\delta S = \int dx^0 d^2x \kappa \partial_i \left[\xi^i \epsilon^{kl} \left(A_0 \partial_k A_l - A_k \partial_0 A_l + A_k \partial_l A_0 \right) \right] \tag{4.53}$$

The integrand is a total derivative and drops to zero when integrated over space. Thus the action is invariant under the general coordinate transformations up to a possible boundary term, as expected. In the presence of boundaries, as in the case of Hall systems, the compensating boundary term to be included can be simplified due to the form of ξ^i. This in particular concerns non-relativistic boosts at a spatial boundary. The action Eq. (4.51) is likewise also gauge invariant. Since the gauge transformations are identical to those in the relativistic case, the boundary terms which restore gauge invariance are the same as those for relativistic backgrounds.

Chapter 5

Dynamical construction of the Newton-Cartan geometry

In this chapter we will consider a detailed description of the Newton-Cartan (NC) geometry in $3+1$ dimensions. Following this we demonstrate how one can construct the NC geometry using the fields introduced during the localization of Galilean symmetry in Section 3.3.

5.1 A brief review of the Newton-Cartan background

The NC background is Cartan's spacetime formulation of the classical Newtonian theory of gravity. It is a classical spacetime with a non-relativistic smooth differentiable manifold 'M' which contains a degenerate inverse spatial metric '$h^{\mu\nu}$' and a degenerate temporal 1-form 'τ_μ' satisfying the following relations,

$$\nabla_\rho h^{\mu\nu} = 0 \qquad \nabla_\mu \tau_\nu = 0$$
$$h^{\mu\nu}\tau_\mu = 0 \tag{5.1}$$

where '∇_μ' is the covariant derivative associated with a connection Γ on the manifold 'M'. (M, h, τ, ∇) is known in the literature as the NC structure of the spacetime. The relations Eq. (5.1) are the compatibility and orthogonality relations. We can define the temporal degenerate metric in terms of the one-form,

$$\tau_{\mu\nu} = \tau_\mu \tau_\nu \tag{5.2}$$

As explained in detail in [13] such a structure serves as the basis for a classical theory of motion in the following way. Given any globally defined and nowhere vanishing vector ξ^μ, the one form 'τ_μ' assigns a temporal length $(\tau_{\mu\nu}\xi^\mu\xi^\nu)^{\frac{1}{2}}$. This allows us to distinguish between timelike and spacelike vectors, depending on whether the temporal length is positive or zero. Likewise, a smooth curve will be timelike if its tangent vectors (e^μ) are timelike at every point. Note that this curve is future directed as its tangent vector always satisfies $\tau_\mu e^\mu > 0$. The orthogonality relation in Eq. (5.1) implies that the subspace of the spacelike vectors is 3-dimensional. The metricity condition of 'τ_μ' indicates that in the absence of torsion the one-form is closed ($\nabla_{[\mu}\tau_{\nu]} = 0$). Thus τ_μ in this case is locally exact and can be expressed in terms of a global time function ($\tau_\mu = \nabla_\mu t$). Given any time function 't' and a timelike curve γ with tangent field e^μ, the temporal length of γ depends only on the endpoints of the curve. This implies that we have a well-defined, path-independent notion of temporal distance between points. Particle trajectories thus follow timelike curves which can be parametrized by 'τ_μ'.

These facts allow the simply connected manifold 'M' to be decomposed into instantaneous 3-d spacelike hypersurfaces 'Σ_t' at constant time 't'. These hypersurfaces satisfy the Frobenius condition $\tau_{[\mu}\nabla_\nu\tau_{\lambda]} = 0$, and as such for any spacelike vector p^μ we have $p^\mu\tau_\mu = 0$. In general, the metricity condition for τ_μ does not necessarily imply that we can define $\tau = \nabla t$. This is particularly the case when $\partial_{[\mu}\tau_{\nu]} \neq 0$, which for the NC background implies the existence of a non-vanishing torsion tensor. We will address this possibility towards the end of the present section. For the moment, we note that in every case $p^\mu\tau_\mu = 0$ is locally true. Given that $h^{\mu\nu}$ and τ_μ are degenerate, their inverses do not exist. Formally, we can define a generalized inverse for the temporal 1-form, τ^μ, such that

$$\tau^\mu\tau_\mu = 1 \tag{5.3}$$

There exists a class of τ^μ which satisfy the above relation, with respect to which we can further define a spatial metric, $h_{\mu\nu}$, that satisfies the following relations

$$\begin{aligned} h_{\mu\nu}\tau^\mu &= 0 \\ \delta^\mu_\nu &= h^{\mu\lambda}h_{\lambda\nu} + \tau^\mu\tau_\nu \end{aligned} \tag{5.4}$$

Here $h^{\mu\lambda}h_{\lambda\nu} = P^\mu_\nu$ is the projection operator of the NC background. The vector field 'τ^μ' can be identified as the four velocity associated with the timelike curve. The corresponding four acceleration ($a^\mu = \tau^\nu\nabla_\nu\tau^\mu$) is spacelike ($\tau_\mu a^\mu = 0$) following the metricity condition of the one-form τ_μ. Thus the four-velocity is timelike and four-acceleration is spacelike. If the particle is massive the acceleration satisfies the equation of motion,

$$F^\mu = ma^\mu \tag{5.5}$$

where F^μ is a spacelike vector field representing the net force acting on the particle. The spatial length of a vector ξ^μ cannot be considered as $(h_{\mu\nu}\xi^\mu\xi^\nu)^{\frac{1}{2}}$ because $\nabla_\rho h_{\mu\nu} \neq 0$. However in this regard $h^{\mu\nu}$ can be used to assign a spatial length to any spacelike vector. For a spacelike vector p^μ the spatial length will be $(h^{\mu\nu}\chi_\mu\chi_\nu)^{\frac{1}{2}}$ where $h^{\mu\nu}\chi_\nu = p^\mu$. All indices of the NC background are raised using the metric $h^{\mu\nu}$.

There exists a covariant derivative which is metric compatible with both the metrics. A direct consequence of this is that the resultant connection is not uniquely determined by these metrics alone. This allows the NC background to geometrically capture the presence of external forces [19]. With all these considerations, a linear symmetric connection which satisfies the metricity conditions given in Eq. (5.1) has a general form given by

$$\begin{aligned}\Gamma^{\rho}{}_{\nu\mu} &= \tau^{\rho}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\Big(\partial_{\mu}h_{\sigma\nu} + \partial_{\nu}h_{\sigma\mu} - \partial_{\sigma}h_{\mu\nu}\Big) + h^{\rho\lambda}\tau_{(\mu}K_{\nu)\lambda} \\ &= \Gamma'^{\rho}_{\nu\mu} + h^{\rho\lambda}\tau_{(\mu}K_{\nu)\lambda} \end{aligned} \tag{5.6}$$

$\Gamma'^{\rho}_{\nu\mu}$ in Eq. (5.6) represents the inertial part of the connection, while the full connection $\Gamma^{\rho}{}_{\nu\mu}$ contains additional non-inertial forces (generating from the Newtonian potential) through the term $K_{\lambda\mu}$ [26].

Given the symmetric connection Eq. (5.6), one can construct the Riemann tensor in the usual way,

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\lambda} = R^{\lambda}_{\sigma\mu\nu}V^{\sigma} \tag{5.7}$$

Note that for a symmetric NC connection, the following relations hold,

$$\tau_{\rho}R^{\rho}_{\sigma\mu\nu} = 0, \quad R^{\lambda}_{\sigma(\mu\nu)} = 0, \quad R^{\lambda}_{[\sigma\mu\nu]} = 0, \quad R^{(\lambda\sigma)}{}_{\mu\nu} = 0 \tag{5.8}$$

The theory considered thus far is completely general. If in addition the Galilean connection has to possess the correct Newtonian limit of the connection of a Riemannian manifold, then the following additional condition known as *Trautman's condition* is required

$$R^{\lambda}{}_{\sigma}{}^{\mu}{}_{\nu} = R^{\mu}{}_{\nu}{}^{\lambda}{}_{\sigma} \tag{5.9}$$

This condition is equivalent to requiring that $dK = 0$, which implies that

$$K_{\lambda\mu} = 2\partial_{[\lambda}A_{\mu]} \tag{5.10}$$

where A_{μ} is at this stage an arbitrary 1-form. If we now define

$$\phi = \tau^{\mu}A_{\mu}\,, \qquad h^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi = 4\pi\rho \tag{5.11}$$

then we can use Eq. (5.8) and Eq. (5.10) to demonstrate that the Ricci tensor satisfies,

$$R_{\mu\nu} = 4\pi\rho\tau_{\mu}\tau_{\nu} \tag{5.12}$$

which is the correct Newtonian limit of Einstein's equations. Eq. (5.11) indicates that 'ρ' is the mass density involved in Poisson's equation. We further note that we can now provide the following equivalent definition for the acceleration defined earlier

$$a^{\mu} = \tau^{\nu}\nabla_{\nu}\tau^{\mu} = \tau^{\nu}K_{\nu\rho}h^{\rho\mu} \tag{5.13}$$

The NC spacetime in this form is described by the quintuplet $(M, h^{\mu\nu}, \tau_{\mu}, \tau^{\nu}, A_{\mu})$, which implicitly considers the Newtonian connection.

Non-relativistic spacetimes also do not have a preferred vector field τ^μ and this imposes an additional symmetry-invariance under the following Milne boost [7,8],

$$\tau^\mu \to \tau^\mu + h^{\mu\nu} k_\nu \tag{5.14}$$

where $k^\mu \tau_\mu = 0$. To satisfy the orthogonality relations between τ^μ and $h_{\mu\nu}$, as well as retain the invariance of the connection Eq. (5.6) under Milne boost, $h_{\mu\nu}$ and A_μ transform accordingly.

$$\begin{aligned} h_{\mu\nu} &\to h_{\mu\nu} - (\tau_\mu P^\rho_\nu + \tau_\nu P^\rho_\mu) k_\rho + \tau_\mu \tau_\nu h^{\rho\sigma} k_\rho k_\sigma \\ A_\mu &\to A_\mu + P^\nu_\mu k_\nu - \frac{1}{2} \tau_\mu h^{\nu\rho} k_\nu k_\rho \end{aligned} \tag{5.15}$$

The second relation of Eq. (5.15) is valid only when the one-form τ_μ is closed. In particular, the NC connection which involves torsion is not simultaneously U(1) invariant and Milne invariant [7].

The Milne transformations of the NC structure is the closest analogue one has of the "shift symmetry" involved in the ADM formalism for general relativistic backgrounds. While the NC structure is quite similar to the ADM decomposition, there are subtle differences which we will now discuss. In the ADM formulation of general relativity, given a non-degenerate spacetime metric $g_{\mu\nu}$, we choose a hypersurface with induced spatial metric $h_{\mu\nu}$ and normal n^μ to the hypersurface.

$$g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \tag{5.16}$$

The normal is associated with the choice of time following $n_\mu = \nabla_\mu t$. n^μ satisfies the orthogonality relation with the induced metric $h_{\mu\nu}$. Both n_μ and $h_{\mu\nu}$ are degenerate when considered as spacetime fields, but are not so within the projective formalism. In other words, by labelling spacetime coordinates by Greek indices and spatial coordinates by Latin indices, $h_{\mu\nu}$ is degenerate while h_{ij} is not. Given the ADM decomposition, we are free to choose an arbitrary vector t^μ as our choice of time which need not be hypersurface orthogonal. This vector can be decomposed in the following way

$$t^\mu = N n^\mu + N^\mu \tag{5.17}$$

where N and N^μ are called the lapse and spatial shift. The choice of time is thus characterized by the lapse and shift variables, where the latter provides a freedom in the choice of time for any given foliation of the background. Thus the gauge variables of the ADM formalism are $(h^{\mu\nu}, N, N^\mu)$. In contrast, the NC geometry involves the gauge variables $(h_{\mu\nu}, \tau^\mu, A_\mu)$. We note further that the spacetime metric $g_{\mu\nu}$ satisfies the metricity condition on relativistic backgrounds and not the induced metric $h_{\mu\nu}$ or n_ν. For the NC geometry, while we can always define the non-degenerate "effective metric" $\gamma_{\mu\nu} = h_{\mu\nu} + \tau_\mu \tau_\nu$ [51], it does not satisfy the metricity condition. The metrics $h^{\mu\nu}$ and τ_μ are those which satisfy the metricity condition leading to the form of the connection and the introduction of an additional gauge field A_μ in the NC structure. Nevertheless, the form of the projection operator in both cases take the same form (the second equation of Eq. (5.4)) [1]. This helps in defining the same covariant measure of the background. This measure follows from the determinant of the metric $\mathcal{G}_{\mu\nu}$

$$\mathcal{G} = \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\mathcal{G}_{\mu\alpha}\mathcal{G}_{\nu\beta}\mathcal{G}_{\rho\gamma}\mathcal{G}_{\sigma\delta} \tag{5.18}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol and $\mathcal{G}$ represents the non-degenerate metric of the spacetime. In the relativistic case $\mathcal{G}_{\mu\nu} = g_{\mu\nu}$ while in the NC case we have $\mathcal{G}_{\mu\nu} = \gamma_{\mu\nu}$. This leads to the following results due to Eq. (5.4)

$$\begin{aligned} \sqrt{g} &= N\sqrt{h} \\ \sqrt{\gamma} &= \sqrt{h} \end{aligned} \tag{5.19}$$

The second equation of Eq. (5.19) is valid for NC backgrounds without torsion and follows from the fact that the lapse for the NC background is always unity ($\tau_\mu\tau^\mu = 1$). As noted in [51] $\sqrt{\gamma} = \sqrt{h}$ is both metric compatible and Milne invariant despite $\gamma_{\mu\nu}$ being neither.

We now turn our attention to the dynamics of fields on the NC background, and of the background itself. We will consider observers who move along τ^μ, which are the comoving observers of the background, as well as those moving along the Milne invariant $\tau^\mu - h^{\mu\nu}A_\nu$. In considering quantum fields on curved backgrounds it is desirable to consider foliations with respect to the Killing vectors of the background, as this ensures that only the fields evolve in going from one time slice to another. Before addressing this in detail, we note that while τ^μ is orthogonal to $h_{\mu\nu}$, it does not satisfy the metricity condition. As such, non-trivial relations exist between $h_{\mu\nu}$ and τ^μ. From Eq. (5.4) we can obtain the variation of $h_{\mu\nu}$ as,

$$\delta h_{\mu\nu} = -2h_{\rho(\mu}\tau_{\nu)}\delta\tau^\rho \tag{5.20}$$

In a similar manner the covariant derivative on $h_{\mu\nu}$ will act in the following way,

$$\nabla_\gamma h_{\mu\nu} = -2h_{\rho(\mu}\tau_{\nu)}\nabla_\gamma\tau^\rho \tag{5.21}$$

These relations will now be used to understand the dynamics of fields on the NC background. A covariant definition of time evolution involves the Lie derivative with respect to the time vector t^μ. Let us first consider the comoving observer $t^\mu = \tau^\mu$. It trivially follows that,

$$\pounds_\tau \tau^\mu = 0 \tag{5.22}$$

and due to Eq. (5.1), we also note that

$$\pounds_\tau \tau_\mu = 0 \tag{5.23}$$

[1]This is the case for relativistic metrics with a Riemannian signature. In the Lorentzian signature, the projection operators agree up to a sign on the temporal part. For comparison with the NC background, we assume the relativistic background has a Riemannian signature.

Thus the Lie derivatives of the temporal one form and vector both vanish. However, the Lie derivatives of the spatial metrics is less trivial. For the metric $h^{\mu\nu}$, we find the non-vanishing expression

$$\begin{aligned}\dot{h}^{\mu\nu} &= \pounds_\tau h^{\mu\nu} = \tau^\rho \nabla_\rho h^{\mu\nu} - h^{\mu\rho}\nabla_\rho \tau^\nu - h^{\nu\rho}\nabla_\rho \tau^\mu \\ &= -2h^{\rho(\mu}\nabla_\rho \tau^{\nu)} \end{aligned} \tag{5.24}$$

Contracting Eq. (5.24) with τ_μ, we find

$$\tau_\mu \pounds_\tau h^{\mu\nu} = 0 \tag{5.25}$$

This implies that $\pounds_\tau h^{\mu\nu}$ is spatial, whose trace is given by,

$$h_{\mu\nu}\pounds_\tau h^{\mu\nu} = -2\nabla_\mu \tau^\mu \tag{5.26}$$

The Lie derivative of the covariant spatial metric is as follows,

$$\pounds_\tau h_{\mu\nu} = \tau^\rho \nabla_\rho h_{\mu\nu} + h_{\mu\rho}\nabla_\nu \tau^\rho + h_{\nu\rho}\nabla_\mu \tau^\rho \tag{5.27}$$

Like the contravariant spatial metric, the Lie derivative of the covariant metric is also spatial with the same trace, up to a sign.

$$\tau^\mu \pounds_\tau h_{\mu\nu} = 0, \quad h^{\mu\nu}\pounds_\tau h_{\mu\nu} = 2\nabla_\rho \tau^\rho \tag{5.28}$$

It can now be noted that if τ^μ satisfies the Killing equation, the Lie derivative acting on the spatial metrics will vanish. In this case we can consider matter fields coupled to the NC geometry as an external, non-dynamical classical background. As non-relativistic theories require the conservation of matter, we further consider the following Lie derivative of the NC gauge field,

$$\pounds_\tau A_\nu = \tau^\nu K_{\nu\mu} + \nabla_\mu(\tau^\nu A_\nu) \tag{5.29}$$

Using Eq. (5.11) and Eq. (5.13), we see that contracting the right hand side of Eq. (5.29) with τ^μ gives $\pounds_\tau \phi$, while contracting it with $h^{\mu\alpha}$ results in $a^\alpha + \nabla^\alpha \phi$. We thus see that if the following equations are satisfied

$$\pounds_\tau \phi = 0\,, \quad a^\mu = -h^{\mu\alpha}\nabla_\alpha \phi \tag{5.30}$$

then $\pounds_\tau A_\mu = 0$. The conditions in Eq. (5.30) are nothing but the Newtonian limit of a General Relativistic background. Thus we can consider the curved background as a fixed classical background for NRFTs when τ^μ is Killing and A_μ satisfies the Newtonian potential conditions provided in Eq. (5.30).

The above discussion can be extended to a Milne invariant choice of time, as in $t^\mu = \tau^\mu - h^{\mu\nu}A_\nu$. In this case a fixed curved background results when τ^μ and $h^{\mu\nu}A_\nu = A^\mu$ are mutually commuting Killing vector fields ($\pounds_\tau A^\mu = 0\,; \pounds_A \tau^\mu = 0$). All the Lie derivatives of the NC fields considered above now also vanish when the Lie derivative with respect to A^μ is considered.

We will now touch upon the particular form of the torsion tensor of the NC background, which will be important to recover as a consequence of the vierbeins introduced through the localization procedure. Its form and properties on the NC background is constrained by the metricity relations, which as a consequence differ substantially from those of relativistic backgrounds.

When the metric τ_μ is not closed, one finds the following relation,

$$\partial_{[\mu}\tau_{\nu]} = \frac{T^{\rho}_{\nu\mu}}{2}\tau_\rho \tag{5.31}$$

where $T^{\rho}{}_{\nu\mu} = 2\Gamma^{\rho}_{[\nu\mu]}$ is the torsion tensor. Thus unlike the relativistic case, here the torsion tensor cannot be specified independently of the metric. As Eq. (5.31) further demonstrates, since the clock form is no longer closed we lose the notion of absolute time for the spatial hypersurface. From the above equation, in [55] the torsion tensor was considered to have the following form,

$$T^{\lambda}_{\mu\nu} = 2\tau^{\lambda}\partial_{[\mu}\tau_{\nu]} \tag{5.32}$$

thereby appearing to have only a temporal component. This form of the NC torsion has found certain applications in non-relativistic holography [52, 53] and condensed matter systems [54]. However in [56] it is shown that the general torsion tensor can have a spatial component. Additional properties of the torsion tensor and its general form will be considered in the following chapter concerning the scale covariant NC background, where the torsion tensor has particularly important dynamical consequences.

5.2 Construction of Newton-Cartan geometry from geometrical interpretation of GGT

Similar to the Poincaré gauge theory which leads to the Einstein-Cartan spacetime [12] the Galilean gauge theory will be shown to reproduce the NC spacetime. Thus one major application of the localization procedure is the construction of the NC geometry through a specific identification of the fields introduced during the localization of Galilean symmetry. A four dimensional manifold can be defined with two coordinate systems - local and global, such that at every global coordinate point there is a local coordinate system. The previously introduced field, $\Sigma_\alpha{}^\mu$, was interpreted as the vierbein in Section 3.4.1 which maps the global and local frames. It was demonstrated in [11] that the 4-d manifold endowed with $\Sigma_\alpha{}^\mu$ and its inverse $\Lambda_\mu{}^\alpha$ had the features of the NC geometry. We will discuss this in the following.

We identify $\Sigma_\alpha{}^\mu$ as the vierbein fields. Then the inverse vierbein $\Lambda_\mu{}^\alpha$ satisfies,

$$\Sigma_\alpha{}^\mu\Lambda_\mu{}^\beta = \delta^\beta_\alpha, \quad \Sigma_\alpha{}^\mu\Lambda_\nu{}^\alpha = \delta^\mu_\nu \tag{5.33}$$

The degenerate inverse spatial metric of rank 3 can be defined as,

$$h^{\mu\nu} = \Sigma_a{}^{\mu}\Sigma_b{}^{\nu}\delta^{ab} \tag{5.34}$$

where the spatial component (h^{ij}) was already defined in Section 3.4.1. The temporal one-form of rank 1 can also be defined in terms of the inverse vierbein field $\Lambda_\mu{}^0$.

$$\tau_\mu = \Lambda_\mu{}^0 \quad (\Lambda_k{}^0 = 0, \Lambda_0{}^0 \neq 0) \tag{5.35}$$

With these definitions, Eq. (3.23) and Eq. (3.25) in addition leads to the following variations of $h^{\mu\nu}$ and τ_μ

$$\delta_0 h^{\mu\nu} = -\xi^\rho\partial_\rho h^{\mu\nu} + h^{\rho\nu}\partial_\rho\xi^\mu + h^{\rho\sigma}\partial_\sigma\xi^\mu$$

$$\delta_0\tau_\mu = -\tau_\mu\partial_0\xi^0 - \xi^0\partial_0\tau_\mu \tag{5.36}$$

Using these relations it is easy to show that the inverse spatial metric and one-form have the correct tensorial properties,

$$h^{\mu\nu}\left(x'\right) = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}h^{\rho\sigma}(x) \tag{5.37}$$

and

$$\tau_\mu\left(x'\right) = \frac{\partial x^\rho}{\partial x'^\mu}\tau_\rho(x) \tag{5.38}$$

The explicit structure of gauge fields 'B' introduced in Section 3.3 can be given by,

$$\begin{aligned} B_k &= B_k^{ab}\lambda_{ab} + B_k^{a0}\lambda_a \\ B_t &= B_t^{ab}\lambda_{ab} + B_t^{a0}\lambda_a \end{aligned} \tag{5.39}$$

where λ_{ab} and λ_a are respectively the generators of rotations and Galilean boosts, B_μ^{ab} are the spin coefficients. Compared to Poincaré case, here $B_\mu{}^{\alpha\beta}$ splits into spatial and temporal part $(B_\mu{}^{ab}, B_\mu{}^{a0})$ [29]. $B_\mu{}^{ab}$ are antisymmetric in indices a, b. The expression for the generator of the Galilean boost is given by $\lambda_a = mx_a$.

The affine connection $\Gamma^\rho_{\nu\mu}$ will be introduced through the vierbein postulate,

$$\nabla_\mu\Lambda_\nu{}^\alpha = \partial_\mu\Lambda_\nu{}^\alpha - \Gamma^\rho_{\nu\mu}\Lambda_\rho{}^\alpha + B^\alpha_{\mu\beta}\Lambda_\nu{}^\beta = 0\,. \tag{5.40}$$

$B^\alpha_{\mu\beta}$ are the spin coefficients introduced in Eq. (5.39). For $\alpha = 0$ we find,

$$\nabla_\mu\Lambda_\nu{}^0 = \partial_\mu\Lambda_\nu{}^0 - \Gamma^\rho_{\nu\mu}\Lambda_\rho{}^0 + B^0_{\mu\beta}\Lambda_\nu{}^\beta = 0\,. \tag{5.41}$$

As $B_\mu{}^{0\beta}$ vanishes for Galilean transformation, it implies,

$$\partial_\mu\Lambda_\nu{}^0 - \Gamma^\rho_{\nu\mu}\Lambda_\rho{}^0 = 0 \tag{5.42}$$

Therefore we reproduce the metricity condition for τ_μ,

$$\nabla_\mu \tau_\nu = 0 \tag{5.43}$$

The inverse spatial metric $h^{\mu\nu}$ can be shown to satisfy the metricity condition. From Eq. (5.33) and Eq. (5.40) one can derive,

$$\partial_\mu {\Sigma_\delta}^\sigma - {B_\mu}^\beta{}_\delta {\Sigma_\beta}^\sigma = -\Gamma^\sigma_{\nu\mu} {\Sigma_\delta}^\nu \tag{5.44}$$

Considering $\delta = a, \beta = b$ we get,

$$\partial_\mu {\Sigma_a}^\sigma - {B_\mu}^b{}_a {\Sigma_b}^\sigma = -\Gamma^\sigma_{\nu\mu} {\Sigma_a}^\nu \tag{5.45}$$

Multiplying ${\Sigma_a}^\rho$ to Eq. (5.45) gives,

$${\Sigma_a}^\rho \partial_\mu {\Sigma_a}^\sigma - {B_\mu}^b{}_a {\Sigma_a}^\rho {\Sigma_b}^\sigma = -\Gamma^\sigma_{\nu\mu} {\Sigma_a}^\rho {\Sigma_a}^\nu \tag{5.46}$$

Then we interchange the indices ρ, σ,

$${\Sigma_a}^\sigma \partial_\mu {\Sigma_a}^\rho - {B_\mu}^b{}_a {\Sigma_a}^\sigma {\Sigma_b}^\rho = -\Gamma^\rho_{\nu\mu} {\Sigma_a}^\sigma {\Sigma_a}^\nu \tag{5.47}$$

Adding Eq. (5.46) with Eq. (5.47) and using the antisymmetric property of ${B_\mu}^{ab}$ leads to,

$$\nabla_\mu h^{\mu\nu} = 0 \tag{5.48}$$

Thus we can conclude that our constructions of $h^{\mu\nu}$ Eq. (5.34) and $\tau_\mu (Eq.$ (5.35) satisfy the metric compatibility conditions.

We can also consider the "inverses" of the metrics by defining $h_{\mu\nu}$ and τ^μ as,

$$h_{\nu\rho} = {\Lambda_\nu}^a {\Lambda_\rho}^a \tag{5.49}$$

and

$$\tau^\rho = {\Sigma_0}^\rho. \tag{5.50}$$

Using Eq. (5.34) and Eq. (5.35) we immediately get,

$$\begin{aligned} h^{\mu\nu}\tau_\nu &= {\Sigma_a}^\mu {\Sigma_a}^\nu {\Lambda_\nu}^0 \\ &= {\Sigma_a}^\mu \delta^0_a \\ &= 0 \end{aligned} \tag{5.51}$$

Also the identifications Eq. (5.50) and Eq. (5.35) show that

$$\tau^\mu \tau_\mu = 1.$$

From the definitions Eq. (5.49) and Eq. (5.50) we find

$$\begin{aligned} h_{\mu\nu}\tau^\nu &= {\Lambda_\mu}^a {\Lambda_\nu}^a {\Sigma_0}^\nu \\ &= {\Lambda_\mu}^a \delta^a_0 \\ &= 0 \end{aligned} \tag{5.52}$$

We can also demonstrate that the projection relation is satisfied

$$h^{\mu\lambda}h_{\lambda\nu} = \Sigma_a{}^{\mu}\Lambda_\nu{}^a = \delta^\mu_\nu - \Sigma_0{}^{\mu}\Lambda_\nu{}^0 \\ = \delta^\mu_\nu - \tau^\mu\tau_\nu. \tag{5.53}$$

This completes the verification of all the orthogonality and projection relations of the NC background, which has followed directly from the constructs of the localisation procedure.

The connection $\Gamma^\rho_{\nu\mu}$ defined in Eq. (5.40) can also be cast in the general form of the NC background. We can write from Eq. (5.40),

$$\Gamma^\rho_{\nu\mu} = \partial_\mu\Lambda_\nu{}^\alpha\Sigma_\alpha{}^\rho + B^\alpha{}_{\mu\beta}\Lambda_\nu{}^\beta\Sigma_\alpha{}^\rho \tag{5.54}$$

Assuming that the connection is symmetric Eq. (5.54) can be written as,

$$\Gamma^\rho_{\nu\mu} = \frac{1}{2}[\Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\mu\nu}] \\ = \frac{1}{2}[\partial_\mu\Lambda_\nu{}^0\Sigma_0{}^\rho + \partial_\nu\Lambda_\mu{}^0\Sigma_0{}^\rho + \partial_\mu\Lambda_\nu{}^a\Sigma_a{}^\rho + \partial_\nu\Lambda_\mu{}^a\Sigma_a{}^\rho \\ + B^a{}_{\mu 0}\Lambda_\nu{}^0\Sigma_\alpha{}^\rho + B^a{}_{\nu 0}\Lambda_\mu{}^0\Sigma_a{}^\rho + B^a{}_{\mu b}\Lambda_\nu{}^b\Sigma_\alpha{}^\rho + B^a{}_{\nu b}\Lambda_\mu{}^b\Sigma_a{}^\rho] \tag{5.55}$$

Using $\Sigma_a{}^\rho = h^{\rho\sigma}\Lambda_\sigma{}^a$ (which follows from Eq. (5.34)), Eq. (5.35) and Eq. (5.50), the above expression will take the form as,

$$\Gamma^\rho_{\nu\mu} = \tau^\rho\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}[\partial_\mu h_{\sigma\nu} - \Lambda_\nu{}^a\partial_\mu\Lambda_\sigma{}^a] + \frac{1}{2}h^{\rho\sigma}[\partial_\nu h_{\sigma\mu} - \Lambda_\mu{}^a\partial_\nu\Lambda_\sigma{}^a] \\ + B^a{}_{0\mu}\Lambda_\nu{}^0\Sigma_\alpha{}^\rho + B^a{}_{0\nu}\Lambda_\mu{}^0\Sigma_a{}^\rho + B^a{}_{\mu b}\Lambda_\nu{}^b\Sigma_\alpha{}^\rho + B^a{}_{\nu b}\Lambda_\mu{}^b\Sigma_a{}^\rho \tag{5.56}$$

Exploiting the symmetricity of $\Gamma^\rho_{\nu\mu}$ we can write,

$$\frac{1}{2}h^{\rho\sigma}[-\Lambda_\nu{}^a\partial_\mu\Lambda_\sigma{}^a - \Lambda_\mu{}^a\partial_\nu\Lambda_\sigma{}^a] = -\partial_\sigma h_{\mu\nu} - B^a{}_{\mu b}\Lambda_\nu{}^b\Sigma_a{}^\rho - B^a{}_{\nu b}\Lambda_\mu{}^b\Sigma_a{}^\rho \tag{5.57}$$

Using Eq. (5.57) we obtain the form of the connection from Eq. (5.56),

$$\Gamma^\rho{}_{\nu\mu} = \tau^\rho\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big) + h^{\rho\lambda}\tau_{(\mu}K_{\nu)\lambda} \tag{5.58}$$

where the two form K is defined as,

$$h^{\rho\lambda}\tau_{(\mu}K_{\nu)\lambda} = \frac{1}{2}h^{\rho\lambda}[\tau_\mu K_{\nu\lambda} + \tau_\nu K_{\mu\lambda}] \\ = \frac{1}{2}h^{\rho\lambda}[\tau_\mu B^a{}_{0\nu}\Lambda_\lambda{}^a + \tau_\nu B^a{}_{0\mu}\Lambda_\lambda{}^a] \tag{5.59}$$

GGT thus fixes the two-form 'K'. The above procedure can also be used to construct other non-relativistic curved backgrounds. For example, the projectable Horava-Lifshitz background can be constructed using the vierbein fields considered in this chapter [28]. These fields and their interrelations may not be wholly satisfied on backgrounds with additional non-relativistic symmetries. In the next chapter, we will consider this in detail in the context of scale invariant non-relativistic theories.

Chapter 6

Inclusion of scale symmetry in the localization procedure

The results of the previous chapters can be extended by including the anisotropic scale transformation in the localization procedure. We will study the construction of scale covariant NC backgrounds from the localization procedure. This will allow us to further investigate the properties of fluids on scale covariant non-relativistic curved backgrounds, which will be discussed in the following chapter.

6.1 Non-relativistic scale symmetry

In relativistic systems, scale transformations act uniformly on space and time, while in non-relativistic systems they act anisotropically [34] and is well known as 'Lifshitz scaling'. Time gets rescaled 'z' times as compared to the space coordinates, where 'z' is called the dynamical critical exponent. Lifshitz scaling plays an important role in condensed matter systems [36] and gravity models which break local Lorentz invariance like Horava-Lifshitz gravity [5]. The role of this scaling in strongly coupled systems, have been investigated holographically and is also found to be relevant in the description of strange metals [35]. The expression of the scale transformations in time and space coordinates, are given by,

$$t' = e^{zs}t, \quad x^{i'} = e^{s}x^{i} \tag{6.1}$$

where 's' is the parameter of the scale transformations. The infinitesimal transformation takes the following form,

$$x^i \to x^i + sx^i, \quad t \to t + zst \tag{6.2}$$

When $z = 1$ the spacetime symmetry group involves the Lorentz group while for the $z = 2$ case it is the Galilean group. For all other values of z, boost invariance will be explicitly broken.

The generators of Lifshitz symmetry for arbitary 'z' correspond to the time translation, spatial translations and the scale transformations. They are denoted by '$P_0 = \partial_t$', '$P_i = \partial_i$' and 'D' respectively. The explicit form of 'D' is,

$$D = -(zt\partial_t + x^i\partial_i) \tag{6.3}$$

The operators D, P_i and P_0 satisfy the following commutation relations,

$$[D, P_i] = P_i, \quad [D, P_0] = zP_0 \tag{6.4}$$

We are in particular interested in Schrödinger field theory which is invariant under $z = 2$ Lifshitz scaling. The Galilean symmetry with both the scale and special conformal symmetry is known as the 'Schrodinger symmetry'. The corresponding algebra is called the 'Schrödinger algebra' [30], which is a conformal extension of the Bargmann algebra. Another non-relativistic conformal extension known in the literature is that of the Galilean Conformal algebra [57]. The generator for non-relativistic scaling in GCA is,

$$D = -(x^i\partial_i + t\partial_t)$$

In GCA, space and time scale as in the relativistic case and the number of generators are the same as those of the relativistic conformal group. In the following subsections we will only consider the Lifshitz scaling with $z = 2$.

6.2 Localization of scale transformation

The first step of localization involves considering a non-relativistic scale invariant field theory. As mentioned, the Schrödinger complex scalar fields on Euclidean space is invariant under the global infinitesimal scale transformations with $z = 2$,

$$x^i \to x^i + sx^i, \quad t \to t + 2st \tag{6.5}$$

where 's' is the parameter of the scale transformation. For the global case 's' is constant. The action of Schrödinger fields is given by,

$$S = \int dt \int d^3x \left[\frac{i}{2}\left(\phi^*\partial_t\phi - \phi\partial_t\phi^*\right) - \frac{1}{2m}\partial_k\phi^*\partial_k\phi\right] \tag{6.6}$$

This action is invariant under global Galilean transformations, which was discussed in Section 3.3. In addition it can be explicitly shown to be invariant ($\Delta\mathcal{L} = 0$) under Eq. (6.5) in $(3+1)$ dimensions provided the field and its derivatives vary in the following way,

$$\begin{aligned}
\delta_0^S\phi &= -\left(\frac{3}{2}s + \xi^i\partial_i + \xi^0\partial_t\right)\phi \\
\delta_0^S\partial_t\phi &= -\left(\frac{7}{2}s + \xi^i\partial_i + \xi^0\partial_t\right)\partial_t\phi \\
\delta_0^S\partial_k\phi &= -\left(\frac{5}{2}s + \xi^i\partial_i + \xi^0\partial_t\right)\partial_k\phi
\end{aligned} \tag{6.7}$$

where $\xi^0 = 2st$ and $\xi^i = sx^i$. Note that unlike what happens under Galilean transformations, here $\partial_\mu \xi^\mu$ is nonvanishing. On account of Eq. (6.5), $\partial_\mu \xi^\mu = 5s$.

Localization of the scale transformation can now be carried out by allowing the parameters of the transformations to be functions of space and time. The absolute nature of the non-relativistic time coordinate requires the most general local scale transformation to be of the following form,

$$x^i \to x^i + s(x,t)x^i, \quad t \to t + \lambda(t)t \tag{6.8}$$

Note that at the time of local scale transformations, the magnitude of the time rescaling parameter always has to be twice the magnitude of the space rescaling parameter to keep the Schrödinger action invariant. We nevertheless require two parameters due to the distinction of time and space.

We will consider both the Galilean and scale transformations in the localization procedure. Under local scale transformations, the derivatives of ϕ will not vary in accordance with Eq. (6.7). To retain the invariance of the action under both the transformations, additional new fields need to be incorporated such that the derivatives of the field ϕ vary covariantly as in Eq. (6.7). This requires the introduction of gauge covariant derivatives with respect to the global coordinates defined in the following way

$$D_\mu \phi = \partial_\mu \phi + iB_\mu \phi + iC_\mu \phi \tag{6.9}$$

where 'B_μ' was already introduced in Section 3.3 at the time of localizing the Galilean symmetry. Here C_μ is included to localize the scale transformation. Similar to the case of Galilean transformations, the new derivatives 'D_μ' do not transform covariantly. A covariant transformation can be achieved in two steps, as stated in the previous chapter. First, local coordinates will be considered at every space-time point of the global coordinate system to enable a geometric interpretation of the localization prescription. Next, local covariant derivatives will be defined in the similar way,

$$\tilde{D}_\alpha \phi = \Sigma_\alpha{}^\mu D_\mu \phi \tag{6.10}$$

where '$\mu(0, i = 1, 2, 3)$' and '$\alpha(0, a = 1, 2, 3)$' indicates the global and local coordinate indices respectively.

Following the definitions Eq. (6.9), Eq. (6.10) it can be observed that the covariant derivatives $\tilde{D}$ transform covariantly in the following way,

$$\begin{aligned}
\delta_0 \tilde{D}_a \phi &= -\xi^0 \partial_t (\tilde{D}_a \phi) - \xi^i \partial_i (\tilde{D}_a \phi) - imv^i x_i (\tilde{D}_a \phi) - \omega^b{}_a \tilde{D}_b \phi - imv_a \phi \\
&\quad - \left(\frac{3s + \lambda}{2}\right) \tilde{D}_a \phi \\
\delta_0 \tilde{D}_0 \phi &= -\xi^0 \partial_t (\tilde{D}_0 \phi) - \xi^i \partial_i (\tilde{D}_0 \phi) - imv^i x_i (\tilde{D}_0 \phi) + v^b \tilde{D}_b \phi \\
&\quad - \left(\frac{s + 3\lambda}{2}\right) \tilde{D}_0 \phi
\end{aligned} \tag{6.11}$$

provided the additional fields 'B_μ', 'C_μ' and 'Σ' transform according to

$$\begin{aligned}
\delta_0 B_k &= -\xi^\mu \partial_\mu B_k - \partial_k \xi^\mu B_\mu + m\partial_k v^i x_i + m(v_k - \Lambda_k{}^a v_a) \\
\delta_0 C_k &= -\xi^\mu \partial_\mu C_k - \partial_k \xi^\mu C_\mu + \frac{1}{2}\partial_k s \\
\delta_0 \Sigma_a{}^k &= -\xi^\mu \partial_\mu \Sigma_a{}^k + \partial_i \xi^k \Sigma_a{}^i - s\Sigma_a{}^k - \omega^b{}_a \Sigma_b{}^k \\
\delta_0 B_t &= -\xi^\mu \partial_\mu B_t - \partial_t \xi^\mu B_\mu + m\Psi^k \Lambda_k{}^a v_a + m\dot{v}^i x_i \\
\delta_0 C_t &= -\xi^\mu \partial_\mu C_t - \partial_t \xi^\mu C_\mu + \frac{1}{2}\partial_t (s + \lambda) \\
\delta_0 \Sigma_0{}^0 &= -\xi^\mu \partial_\mu \Sigma_0{}^0 + \partial_t \xi^0 \Sigma_0{}^0 \\
\delta_0 \Sigma_0{}^k &= -\xi^\mu \partial_\mu \Sigma_0{}^k + \partial_\mu \xi^k \Sigma_0{}^\mu - \lambda \Sigma_0{}^k + v^b \Sigma_b{}^k
\end{aligned} \tag{6.12}$$

where $\xi^0 = (\epsilon(t) - \lambda(t)t)$, $\xi^i = \left(\eta^i(x,t) - tv^i(x,t) + s(x,t)x^i\right)$. Invariance of the local action requires that the magnitude of time scaling parameter λ be twice of the space scaling parameter s. The inverse of the 'Σ' fields are defined as,

$$\Sigma_\alpha{}^\mu \Lambda_\nu{}^\alpha = \delta^\mu_\nu, \qquad \Sigma_\alpha{}^\mu \Lambda_\mu{}^\beta = \delta^\beta_\alpha \tag{6.13}$$

B_μ are already expressed in terms of spin coefficients and generators in Eq. (5.39). The fields C_μ can be defined as,

$$C_\mu = Db_\mu \tag{6.14}$$

where D is the generator of scale transformations. We can now replace the partial derivatives in the action Eq. (3.11) with these local covariant derivatives to give,

$$\mathcal{L}\left(\phi, \partial_t \phi, \partial_k \phi\right) \to \mathcal{L}'\left(\phi, \tilde{D}_0 \phi, \tilde{D}_a \phi\right)$$

Similar to the Galilean case we have to consider the change in the measure,

$$\Lambda = \frac{1}{\Sigma_0{}^0} \det \Lambda_k{}^a = \det \Lambda_\mu{}^\alpha \tag{6.15}$$

Replacing the partial derivatives with the local covariant derivatives and considering the change in the measure, the Schrödinger action Eq. (6.6) modifies to,

$$S = \int dt \int d^3x \left(\frac{1}{\Sigma_0{}^0} \det \Lambda_k{}^a\right) \left[\frac{i}{2}\left(\phi^* \tilde{D}_0 \phi - \phi \tilde{D}_0 \phi^*\right) - \frac{1}{2m} \tilde{D}_a \phi^* \tilde{D}_a \phi\right]. \tag{6.16}$$

Note that unlike in relativistic theories, the *mass* is not the coefficient of the linear term in the potential here, but enters as a passive parameter in the kinetic term since non-relativistic theories hold in the regime where the energies being dealt with are far less than the (rest) mass. As such, massive scale invariant non-relativistic theories can and do exist.

6.3 Scale covariant Newton-Cartan geometry

As we have seen, the additional inclusion of scale invariance has led to a different result following localization. First, the transformation properties of the gauge fields that were introduced at the time of localization of the Galilean symmetry were modified. Second, the localization procedure brought in additional gauge fields that were required in order to render the action invariant. The gauge fields reduce to those found in the localization of Galilean symmetry when the scale parameter '$s, \lambda \to 0$'. We would thus expect a scale covariant geometry to emerge upon identifying the vierbeins of the manifold. However, this geometric structure should reduce to the NC geometry in the limit of vanishing scale parameters.

We begin by defining the inverse spatial metric as before,

$$h^{\mu\nu} = \Sigma_a{}^{\mu}\Sigma_b{}^{\nu}\delta^{ab} \tag{6.17}$$

The temporal one-form can also be defined in terms of the inverse vierbein field $\Lambda_\mu{}^0$.

$$\tau_\mu = \Lambda_\mu{}^0 \quad (\Lambda_k{}^0 = 0, \Lambda_0{}^0 \neq 0) \tag{6.18}$$

With these definitions, Eq. (6.12) leads to the following variations of $h^{\mu\nu}$ and τ_μ,

$$\delta_0 h^{\mu\nu} = -\xi^\rho\partial_\rho h^{\mu\nu} + h^{\rho\nu}\partial_\rho\xi^\mu + h^{\mu\rho}\partial_\rho\xi^\nu - 2sh^{\mu\nu}$$

$$\delta_0\tau_\mu = -\tau_\mu\partial_0\xi^0 - \xi^0\partial_0\tau_\mu + 2s\tau_\mu \tag{6.19}$$

To obtain a full geometric structure the connection will be introduced following the vierbein postulate, which will also help to explore the metricity condition for this geometry. The vierbein postulate is given as follows,

$$\begin{aligned}\tilde{\nabla}_\mu\Lambda_\nu{}^0 &= \partial_\mu\Lambda_\nu{}^0 - \tilde{\Gamma}^\rho_{\nu\mu}\Lambda_\rho{}^0 + B^0{}_{\mu\beta}\Lambda_\nu{}^\beta + 2b_\mu\Lambda_\nu{}^0 = 0\\ \tilde{\nabla}_\mu\Lambda_\nu{}^a &= \partial_\mu\Lambda_\nu{}^a - \tilde{\Gamma}^\rho_{\nu\mu}\Lambda_\rho{}^a + B^a{}_{\mu\beta}\Lambda_\nu{}^\beta + b_\mu\Lambda_\nu{}^a = 0\end{aligned} \tag{6.20}$$

where $\tilde{\Gamma}^\rho_{\nu\mu}$ is the scale covariant connection and b_μ is the scale gauge field defined in Eq. (6.14). Using the fact that $B^0{}_{\mu\beta}$ vanishes for Galilean transformations, we get the following expression from the first equation of Eq. (6.20),

$$\partial_\mu\Lambda_\nu{}^0 - \tilde{\Gamma}^\rho_{\nu\mu}\Lambda_\rho{}^0 = \tilde{\nabla}_\mu\tau_\nu = -2b_\mu\tau_\nu \tag{6.21}$$

From Eq. (6.21) it is evident that τ_μ does not satisfy the metricity conditions.

The metricity condition for $h^{\mu\nu}$ can similarly be derived from Eq. (6.20). From Eq. (6.13) and the second equation of Eq. (6.20) it can be shown that,

$$\partial_\mu\Sigma_a{}^\sigma - B_\mu{}^b{}_a\Sigma_b{}^\sigma - b_\mu\Sigma_a{}^\sigma = -\tilde{\Gamma}^\sigma_{\nu\mu}\Sigma_a{}^\nu \tag{6.22}$$

By contracting Eq. (6.22) by $\Sigma_a{}^\rho$ and using the antisymmetric property of $B_\mu{}^{ab}$, we find that

$$\tilde{\nabla}_\mu h^{\rho\sigma} = \partial_\mu h^{\rho\sigma} + \tilde{\Gamma}^\rho_{\nu\mu}h^{\nu\sigma} + \tilde{\Gamma}^\sigma_{\nu\mu}h^{\nu\rho} = 2b_\mu h^{\rho\sigma}. \tag{6.23}$$

Explicitly taking $b_\mu \to 0$ in Eq. (6.21) and Eq. (6.23) results in the well known metricity conditions for NC geometry.

The respective inverses of the contravariant metric and temporal one-form will be defined as,

$$h_{\nu\rho} = \Lambda_\nu{}^a \Lambda_\rho{}^a \tag{6.24}$$

and

$$\tau^\rho = \Sigma_0{}^\rho . \tag{6.25}$$

Despite the non-metricity, both the orthogonality and projection relations are satisfied by the scale covariant NC background.

$$\begin{aligned} h^{\mu\nu}\tau_\nu = 0,\ h_{\mu\nu}\tau^\nu = 0, \\ h^{\mu\lambda}h_{\lambda\nu} = \delta^\mu_\nu - \tau^\mu\tau_\nu,\ \tau^\mu\tau_\mu = 1 \end{aligned} \tag{6.26}$$

In the context of the covariant derivative, the explicit form of the connection can be determined. This follows from the vierbein postulate by contracting Eq. (6.20) with $\Sigma_\alpha{}^\sigma$, which gives the following general expression for the connection

$$\tilde{\Gamma}^\rho_{\nu\mu} = \partial_\mu \Lambda_\nu{}^\alpha \Sigma_\alpha{}^\rho + B^\alpha{}_{\mu\beta}\Lambda_\nu{}^\beta \Sigma_\alpha{}^\rho + 2b_\mu \Lambda_\nu{}^0 \Sigma_0{}^\rho + b_\mu \Lambda_\nu{}^a \Sigma_a{}^\rho \tag{6.27}$$

From the metricity condition of τ_μ Eq. (6.21), we find the following relation

$$\partial_{[\mu}\tau_{\nu]} = \frac{\tilde{T}^\rho_{\nu\mu}}{2}\tau_\rho - 2b_{[\mu}\tau_{\nu]} \tag{6.28}$$

where $\tilde{T}^\rho_{\nu\mu} = 2\tilde{\Gamma}^\rho_{[\nu\mu]}$ is the torsion tensor of the scale covariant NC background. Note that, for the NC background the temporal component of the torsion tensor ($T^\rho_{\mu\nu}\tau_\rho$) vanishes if $d\tau = 0$ which can be observed from Eq. (5.31). In including scale transformations the torsion tensor and its components acquire additional constraints due to the inclusion of the dilatation gauge field b_μ. We can infer two important facts from the relation Eq. (6.28). First, due to the presence of the scale term in Eq. (6.27) $\tilde{T}^\rho_{\mu\nu}\tau_\rho \neq 0$ even while $d\tau = 0$. This distinguishes this result from that of the NC background described above. The second implication is that when $\tilde{T}^\rho_{\mu\nu}\tau_\rho = 0$ we have the following condition

$$\partial_{[\mu}\tau_{\nu]} = -2b_{[\mu}\tau_{\nu]} \tag{6.29}$$

This equation relates the gauge field connected with scale transformations and the temporal one-form. In particular, Eq. (6.29) still leads to the Frobenius condition being satisfied, ensuring the existence of spatial hypersurfaces orthogonal to τ_μ. Backgrounds satisfying Eq. (6.29) are known in the literature as the Twistless Torsional NC background (TTNC) [30].

The above cases dealt with different consequences involving the temporal component of the torsion tensor. In general, the spatial component always exists and it will thus be useful to determine the general expression of the torsion tensor for the scale covariant NC background. From the vierbein postulate Eq. (6.20), we have

$$\partial_{[\mu}\Lambda_{\nu]}{}^{\alpha} - \tilde{\Gamma}^{\rho}_{[\nu\mu]}\Lambda_{\rho}{}^{\alpha} + B^{\alpha}_{[\mu|\beta|}\Lambda_{\nu]}{}^{\beta} + 2b_{[\mu}\Lambda_{\nu]}{}^{0}\delta^{\alpha}_{0} + b_{[\mu}\Lambda_{\nu]}{}^{b}\delta^{\alpha}_{b} = 0 \tag{6.30}$$

Contracting with $\Sigma_{\alpha}{}^{\sigma}$ on both sides results in

$$\tau^{\sigma}\partial_{[\mu}\tau_{\nu]} + \Sigma_{a}{}^{\sigma}(\partial_{[\mu}\Lambda^{a}_{\nu]} + B^{a}_{[\mu|\beta|}\Lambda_{\nu]}{}^{\beta}) + b_{[\mu}\tau_{\nu]}\tau^{\sigma} + b_{[\mu}\delta^{\sigma}_{\nu]} = \frac{\tilde{T}^{\sigma}_{\nu\mu}}{2} \tag{6.31}$$

Manipulating the terms in the parenthesis one can write the general torsion tensor as

$$\begin{aligned}\frac{\tilde{T}^{\sigma}_{\nu\mu}}{2} &= \left[\tau^{\sigma}\partial_{[\mu}\tau_{\nu]} + 2b_{[\mu}\tau_{\nu]}\tau^{\sigma}\right] \\ &\quad + \left(\partial_{[\mu}\Lambda_{\nu]}{}^{a} + B^{a}_{[\mu|b}\Lambda_{\nu]}{}^{b} + b_{[\mu}\Lambda_{\nu]}{}^{a}\right) h^{\sigma\gamma}\Lambda_{\gamma}{}^{a} + K_{\gamma[\nu}\tau_{\mu]}h^{\sigma\gamma}\end{aligned} \tag{6.32}$$

where the first line represents the temporal contribution and the second line includes the spatial contribution.

We can now express the connection in terms of the metrics and the gauge field (b_{μ}) defined earlier. Making use of Eq. (6.27), the symmetric part of connection can be written as,

$$\begin{aligned}\tilde{\Gamma}^{\rho}_{\nu\mu} &= \frac{1}{2}[\tilde{\Gamma}^{\rho}_{\nu\mu} + \tilde{\Gamma}^{\rho}_{\mu\nu}] = \frac{1}{2}\left[(\partial_{\mu}\Lambda_{\nu}{}^{0}\Sigma_{0}{}^{\rho} + \partial_{\nu}\Lambda_{\mu}{}^{0}\Sigma_{0}{}^{\rho}) + (\partial_{\mu}\Lambda_{\nu}{}^{a}\Sigma_{a}{}^{\rho} + \partial_{\nu}\Lambda_{\mu}{}^{a}\Sigma_{a}{}^{\rho})\right. \\ &\quad + (B^{a}{}_{\mu 0}\Lambda_{\nu}{}^{0}\Sigma_{a}{}^{\rho} + B^{a}{}_{\nu 0}\Lambda_{\mu}{}^{0}\Sigma_{a}{}^{\rho}) + (B^{a}{}_{\mu b}\Lambda_{\nu}{}^{b}\Sigma_{a}{}^{\rho} + B^{a}{}_{\nu b}\Lambda_{\mu}{}^{b}\Sigma_{a}{}^{\rho}) \\ &\quad \left.+2(b_{\mu}\Lambda_{\nu}{}^{0}\Sigma_{0}{}^{\rho} + b_{\nu}\Lambda_{\mu}{}^{0}\Sigma_{0}{}^{\rho}) + (b_{\mu}\Lambda_{\nu}{}^{a}\Sigma_{a}{}^{\rho} + b_{\nu}\Lambda_{\mu}{}^{a}\Sigma_{a}{}^{\rho})\right]\end{aligned} \tag{6.33}$$

Using $\Sigma_{a}{}^{\rho} = h^{\rho\sigma}\Lambda_{\sigma}{}^{a}$ (which follows from Eq. (6.17), Eq. (6.18) and Eq. (6.25), the above expression will take the following form,

$$\begin{aligned}\tilde{\Gamma}^{\rho}_{\nu\mu} &= \tau^{\rho}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}[\partial_{\mu}h_{\sigma\nu} - \Lambda_{\nu}{}^{a}\partial_{\mu}\Lambda_{\sigma}{}^{a}] + \frac{1}{2}h^{\rho\sigma}[\partial_{\nu}h_{\sigma\mu} - \Lambda_{\mu}{}^{a}\partial_{\nu}\Lambda_{\sigma}{}^{a}] \\ &\quad + \frac{1}{2}(B^{a}{}_{0\mu}\Lambda_{\nu}{}^{0}\Sigma_{a}{}^{\rho} + B^{a}{}_{0\nu}\Lambda_{\mu}{}^{0}\Sigma_{a}{}^{\rho} + B^{a}{}_{\mu b}\Lambda_{\nu}{}^{b}\Sigma_{a}{}^{\rho} + B^{a}{}_{\nu b}\Lambda_{\mu}{}^{b}\Sigma_{a}{}^{\rho}) \\ &\quad + \frac{1}{2}(b_{\mu}\delta^{\rho}_{\nu} + b_{\nu}\delta^{\rho}_{\mu} + b_{\mu}\tau_{\nu}\tau^{\rho} + b_{\nu}\tau_{\mu}\tau^{\rho})\end{aligned} \tag{6.34}$$

Since we are now considering the symmetric part of $\tilde{\Gamma}^{\rho}_{\nu\mu}$, we have

$$\begin{aligned}[-\Lambda_{\nu}{}^{a}\partial_{\mu}\Lambda_{\sigma}{}^{a} - \Lambda_{\mu}{}^{a}\partial_{\nu}\Lambda_{\sigma}{}^{a}] &= (-\partial_{\sigma}h_{\mu\nu} - B^{a}{}_{\mu b}\Lambda_{\nu}{}^{b}\Lambda_{\sigma}{}^{a} - B^{a}{}_{\nu b}\Lambda_{\mu}{}^{b}\Lambda_{\sigma}{}^{a}) \\ &\quad + (b_{\mu}h_{\sigma\nu} + b_{\nu}h_{\sigma\mu} - 2b_{\sigma}h_{\nu\mu})\end{aligned} \tag{6.35}$$

Plugging this expression in Eq. (6.34) the symmetric connection of the scale covariant NC geometry can be written as

$$\begin{aligned}\tilde{\Gamma}^{\rho}_{\nu\mu} &= \tau^{\rho}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}h_{\sigma\nu} + \partial_{\nu}h_{\sigma\mu} - \partial_{\sigma}h_{\mu\nu}\right) + (b_{\mu}\delta^{\rho}_{\nu} + b_{\nu}\delta^{\rho}_{\mu} - b_{\sigma}h^{\rho\sigma}h_{\nu\mu}) \\ &\quad + h^{\rho\lambda}\tau_{(\mu}K_{\nu)\lambda}\end{aligned} \tag{6.36}$$

where the two form K is defined in a similar way as stated in Section 5.2.

$$h^{\rho\lambda}\tau_{(\mu}K_{\nu)\lambda} = \frac{1}{2}h^{\rho\lambda}[\tau_\mu \Lambda_\lambda{}^a B^a{}_{0\nu} + \tau_\nu \Lambda_\lambda{}^a B^a{}_{0\mu}] \tag{6.37}$$

Defining 'K' in this way makes the connection unique. It is evident from Eq. (6.36) that in the limit of vanishing 'b_μ', the expression reduces to that of the NC connection. For completeness, we note that in the presence of torsion Eq. (6.32) the general connection becomes,

$$\begin{aligned}\tilde{\Gamma}^\rho_{\nu\mu} &= \tau^\rho \partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\Big(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\Big) + (b_\mu \delta^\rho_\nu + b_\nu \delta^\rho_\mu - b_\sigma h^{\rho\sigma} h_{\nu\mu}) \\ &\quad + h^{\rho\lambda}\tau_{(\mu}K_{\nu)\lambda} + \frac{1}{2}h^{\rho\sigma}\Big[-\tilde{T}_{\mu\nu\sigma} - \tilde{T}_{\nu\mu\sigma} + \tilde{T}_{\sigma\nu\mu}\Big]\end{aligned} \tag{6.38}$$

where $\tilde{T}_{\sigma\nu\mu} = (h_{\sigma\rho} + \tau_\sigma\tau_\rho)\,\tilde{T}^\rho_{\mu\nu}$ and $\tilde{T}^\sigma_{\nu\mu}$ was defined in Eq. (6.32).

Let us now consider the curvature terms and their properties, again only for the symmetric connection ($\tilde{T}^\sigma_{\mu\nu} = 0$). Our analysis will be considered in n spacetime dimensions for the remainder of this chapter. For convenience, we will write the symmetric connection of the scale covariant NC geometry in the following way,

$$\tilde{\Gamma}^\rho_{\nu\mu} = \Gamma^\rho_{\nu\mu} + (b_\mu \delta^\rho_\nu + b_\nu \delta^\rho_\mu - b_\sigma h^{\rho\sigma} h_{\nu\mu}) \tag{6.39}$$

where $\Gamma^\rho_{\nu\mu}$ represents the usual symmetric NC connection. The Riemann tensor for the symmetric connection in Eq. (6.39) is defined in the usual way,

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]V^\lambda = \tilde{R}^\lambda{}_{\sigma\mu\nu}V^\sigma \tag{6.40}$$

Upon expansion, we find the following result

$$\begin{aligned}\tilde{R}^\lambda{}_{\sigma\mu\nu} &= R^\lambda{}_{\sigma\mu\nu} + 2\nabla_{[\mu}(b_{\nu]}\delta^\lambda_\sigma + \delta^\lambda_{\nu]}b_\sigma - h_{\nu]\sigma}b_\delta h^{\delta\lambda}) + 2\delta^\lambda_{[\mu}(b_{\nu]}b_\sigma - h_{\nu]\sigma}b_\rho b_\sigma h^{\rho\sigma}) \\ &\quad + 2b_\rho h^{\rho\lambda}b_{[\mu}h_{\nu]\sigma} - 2b_\rho\tau^\rho\tau_{[\mu}h_{\nu]\sigma}b_\gamma h^{\gamma\lambda}\end{aligned} \tag{6.41}$$

For the NC background $\tau_\lambda R^\lambda{}_{\sigma\mu\nu} = 0$ allowed us to use $R_{\lambda\sigma\mu\nu} = h_{\lambda\rho}R^\rho{}_{\sigma\mu\nu}$. Unlike $R^\lambda{}_{\sigma\mu\nu}$, $\tilde{R}^\lambda{}_{\sigma\mu\nu}$ do not satisfy the properties given in Eq. (5.8) and Eq. (5.9). We require $\delta^\mu_\lambda \tilde{R}^\lambda{}_{\sigma\mu\nu} = \tilde{R}_{\sigma\nu}$, which implies that one can lower the indices of $\tilde{R}^\lambda{}_{\sigma\mu\nu}$ with the combination $(h_{\mu\nu} + \tau_\mu\tau_\nu)$ and raise with $(h^{\mu\nu} + \tau^\mu\tau^\nu)$. Let us first consider

$$\begin{aligned}\tilde{R}_{\epsilon\sigma\mu\nu} &= (h_{\lambda\epsilon} + \tau_\lambda\tau_\epsilon)\tilde{R}^\lambda_{\sigma\mu\nu} \\ &= R_{\epsilon\sigma\mu\nu} + 2(h_{\epsilon\sigma} + \tau_\epsilon\tau_\sigma)\nabla_{[\mu}b_{\nu]} + 2(h_{\epsilon[\nu}\nabla_{\mu]}b_\sigma + \tau_\epsilon\tau_{[\nu}\nabla_{\mu]}b_\sigma) \\ &\quad - 2\nabla_{[\mu}(h_{\nu]\sigma}b_\epsilon) + 2\tau^\delta\tau_\epsilon\nabla_{[\mu}(h_{\nu]\sigma}b_\delta) + 2h_{\epsilon[\mu}b_{\nu]}b_\sigma + 2\tau_\epsilon\tau_{[\mu}b_{\nu]}b_\sigma \\ &\quad - 2h_{\epsilon[\mu}h_{\nu]\sigma}h^{\gamma\rho}b_\gamma b_\rho - 2\tau_\epsilon\tau_{[\mu}h_{\nu]\sigma}h^{\gamma\rho}b_\gamma b_\rho + 2b_\epsilon b_{[\mu}h_{\nu]}\sigma - 2b_\rho\tau^\rho\tau_\epsilon b_{[\mu}h_{\nu]\sigma} \\ &\quad - 2b_\rho\tau^\rho\tau_{[\mu}h_{\nu]\sigma}b_\epsilon + 2\tau^\gamma\tau^\rho b_\gamma b_\rho\tau_\epsilon\tau_{[\mu}h_{\nu]\sigma}\end{aligned} \tag{6.42}$$

We get the expression for the Ricci tensor $\tilde{R}_{\sigma\nu}$ by contracting Eq. (6.42) with $(h^{\epsilon\mu} + \tau^{\epsilon}\tau^{\mu})$,

$$\begin{aligned}\tilde{R}_{\sigma\nu} &= R_{\sigma\nu} + 2\nabla_{[\sigma}b_{\nu]} - \nabla_{\mu}(h_{\nu\sigma}b_{\epsilon}h^{\epsilon\mu}) + (n-2)[b_{\nu}b_{\sigma} - \nabla_{\nu}b_{\sigma} - h_{\nu\sigma}h^{\gamma\rho}b_{\gamma}b_{\rho}] \\ &\quad - \tau_{\sigma}\nabla_{\nu}(\tau^{\rho}b_{\rho}) + 2b_{\rho}\tau^{\rho}\tau_{(\sigma}b_{\nu)} - (b_{\rho}\tau^{\rho})(b_{\gamma}\tau^{\gamma})(\tau_{\nu}\tau_{\sigma}) \end{aligned} \tag{6.43}$$

This is of course the same result one would get from Eq. (6.41) by setting '$\lambda = \mu$'. Again contracting Eq. (6.43) with $(h^{\sigma\nu} + \tau^{\sigma}\tau^{\nu})$ the following expression of the Ricci scalar is obtained,

$$\begin{aligned}\tilde{R} &= R - h^{\mu\nu}\nabla_{\mu}b_{\nu}(2n-3) - (\tau^{\mu}\nabla_{\mu}(b_{\rho}\tau^{\rho}) - \tau^{\gamma}\tau^{\rho}b_{\gamma}b_{\rho})(n-1) \\ &\quad + (n-2)b_{\sigma}\tau^{\rho}\nabla_{\rho}\tau^{\sigma} - (n-2)^2 h^{\gamma\rho}b_{\gamma}b_{\rho} \end{aligned} \tag{6.44}$$

It is evident from the previous expressions Eq. (6.42)-Eq. (6.44) that the Riemann tensor, Ricci tensor and Ricci scalar are not invariant under the scale transformations and certain symmetries of the NC Riemann tensor are not satisfied by the rescaled counterpart. For instance, Eq. (6.43) reveals that the Ricci tensor is not symmetric. If we require $\tilde{R}_{[\sigma\nu]} = \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 0$, this in turn determines conditions on the 'b_{μ}' fields through which these symmetries are satisfied. In General Relativity this simply leads to the condition that '$b_{\sigma} = \partial_{\sigma}\alpha$', for some scalar field α. Here, apart from this constraint the additional requirement of '$b_{[\mu}\tau_{\nu]} = 0$' needs to be satisfied. As can be noted in Eq. (6.29), this will be satisfied when $d\tau = 0$ in the TTNC background.

This motivates us to find an invariant tensor under the anisotropic scale transformation. From Eq. (6.41) we can construct the Weyl tensor $\tilde{C}^{\lambda}{}_{\sigma\mu\nu}$. It can be observed that the Weyl tensor is invariant under non-relativistic scale transformations which implies,

$$\tilde{C}^{\lambda}{}_{\sigma\mu\nu} = C_{\lambda\sigma\mu\nu} = R_{\lambda\sigma\mu\nu} + 2(h_{\lambda[\mu}S_{\nu]\sigma} + \tau_{\lambda}\tau_{[\mu}S_{\nu]\sigma}) - 2(h_{\sigma[\mu}S_{\nu]\lambda} + \tau_{\sigma}\tau_{[\mu}S_{\nu]\lambda}) \tag{6.45}$$

where $R_{\lambda\sigma\mu\nu}$ is the NC Riemann tensor and $S_{\nu\sigma}$ defines the NC Schouten tensor,

$$S_{\nu\sigma} = \frac{1}{n-2}\left(R_{\sigma\nu} - \frac{1}{2(n-1)}R(h_{\sigma\nu} + \tau_{\sigma}\tau_{\nu})\right) \tag{6.46}$$

In general it may be useful to consider the symmetries of the rescaled Riemann tensor without imposing additional conditions. This will be useful in the treatment of non-ideal conformal fluids on curved backgrounds [58]. In the next chapter we will consider the treatment of non-relativistic ideal fluids on the NC and scale covariant NC background. We will briefly discuss the above point in the context of scale invariant fluids.

Chapter 7

Non-relativistic fluids on curved backgrounds

The aim of this chapter is to elaborate on an important application of the construction of the previous chapter, namely, in the description of non-relativistic fluids. Fields close to equilibrium admit a hydrodynamic description. Within this description the stress tensor and symmetry currents are expressed in a gradient expansion of the fluid variables and the spacetime background. We will first give a detailed description of ideal fluids on the NC background. Following this we develop a Weyl-covariant formalism which simplifies the study of conformal covariant non-relativistic hydrodynamics. In particular we will consider scale invariant fluids. In the last section we will investigate some consequences of scale covariant backgrounds on the response functions of Hall fluids.

7.1 Fluids on the Newton-Cartan background

In the non-relativistic hydrodynamics regime, the basic fluid variables are the local velocity $v^i(x)$ and mass density $\rho(x)$, and they satisfy the following conservation equations,

$$\begin{aligned}
\partial_t\rho + \partial_i(\rho v^i) &= 0 \quad \text{(Continuity equation)}\\
\partial_t(\rho v^i) + \partial_i T^{ij} &= 0 \quad \text{(Momentum conservation equation)}\\
\partial_t\left(\epsilon + \frac{1}{2}\rho\mathbf{v}^2\right) + \partial_i j^i &= 0 \quad \text{(Energy conservation equation)}
\end{aligned} \tag{7.1}$$

where T^{ij}, ϵ and j^i are the stress-energy tensor, energy density and matter current of fluid respectively.

A preliminary study of non-relativistic fluids on the usual NC background was performed in [25, 55]. In this section we review some of the relevant properties of ideal non-relativistic fluids on the NC background. The description of non-relativistic fluids requires a choice of fluid velocity. For this purpose, let us consider the fluid velocity u^μ such that

$$u^\mu \tau_\mu = 1 \tag{7.2}$$

A sensible requirement is that the fluid has no acceleration and is irrotational when considered with respect to the inertial frame of the NC background, i.e.

$$a'^\mu = u^\rho \nabla'_\rho u^\mu = 0, \quad \omega'^{\mu\nu} = h^{\gamma[\mu} \nabla'_\gamma u^{\nu]} = 0 \tag{7.3}$$

where ∇' is the covariant derivative corresponding to the inertial piece of the NC connection Eq. (5.6). The total covariant derivative will act on the fluid velocity u^ν as,

$$\nabla_\mu u^\nu = \nabla'_\mu u^\nu + \frac{1}{2} h^{\nu\lambda} K_{\mu\lambda} + \frac{1}{2} h^{\nu\lambda} K_{\rho\lambda} \tau_\mu u^\rho = \nabla'_\mu u^\nu + h^{\nu\lambda} \tau_{(\mu} K_{\rho)\lambda} u^\rho \tag{7.4}$$

From Eq. (7.3) and Eq. (7.4), it then follows that the fluid variables for the expansion, acceleration, shear and vorticity for a general NC background can be written as,

$$\begin{aligned}
\theta &= \nabla_\mu u^\mu = \nabla'_\mu u^\mu = \theta' \\
a^\nu &= u^\mu \nabla_\mu u^\nu = h^{\nu\lambda} K_{\rho\lambda} u^\rho \\
\sigma^{\mu\nu} &= [h^{\lambda(\mu} \nabla_\lambda u^{\nu)}] - \frac{\theta}{n-1} h^{\mu\nu} = [h^{\lambda(\mu} \nabla'_\lambda u^{\nu)}] - \frac{\theta}{n-1} h^{\mu\nu} = \sigma'^{\mu\nu} \\
\omega^{\mu\nu} &= [h^{\lambda[\mu} \nabla_\lambda u^{\nu]}] = \omega'^{\mu\nu} = 0
\end{aligned} \tag{7.5}$$

Thus apart from the acceleration, all other quantities to describe the fluid are invariant in going from an inertial to a non-inertial frame. In addition to these quantities, the description of a fluid requires a definition of the stress energy tensor and other matter currents of the theory. Since the NC background contains two degenerate metrics $(h^{\mu\nu}, \tau_\mu)$ and additional gauge fields $(h_{\mu\nu}, \tau^\mu, A_\mu)$, these definitions should follow from a careful variation of the action. The most general variation of the action, which leaves the connection invariant is given by

$$0 = \delta S = \int \sqrt{h} d^4x [-\frac{1}{2} P_{\mu\nu} \delta h^{\mu\nu} + Q^\mu \delta \tau_\mu + J^\mu \delta A_\mu + R_\mu \delta \tau^\mu] \tag{7.6}$$

where $P_{\mu\nu}, Q^\mu, J^\mu$ and R_μ will be identified with the physical stress tensor, energy current, mass conservation current and momentum current respectively. Two of these variations correspond to non-gauge variables, i.e. $\delta h^{\mu\nu}$ and $\delta \tau_\mu$, which are the variations of the given inverse spatial metric and temporal 1-form. Setting these variations to vanish provides the contributions from the pure gauge variables A_μ and τ^μ. Eq. (7.6) then reduces to,

$$\delta S = \int \sqrt{h} d^4x [J^\mu \delta A_\mu + R_\mu \delta \tau^\mu] \tag{7.7}$$

We can simplify Eq. (7.7) further by using the following properties of $K_{\lambda\mu}$,

$$K_{\lambda\mu} = -2h_{\nu[\lambda}\nabla_{\mu]}\tau^{\nu}, \quad \delta K_{\lambda\mu} = 2\nabla_{[\lambda}h_{\mu]\nu}\delta\tau^{\nu} \tag{7.8}$$

where we have used Eq. (5.4). Following Eq. (5.10) and Eq. (7.8) we get,

$$\delta A_{\mu} = h_{\mu\rho}\delta\tau^{\rho} + \partial_{\mu}\chi \tag{7.9}$$

where $\partial_{\mu}\chi$ represents the U(1) transformation of A_{μ}. Using the expression of δA_{μ} from Eq. (7.9) the action Eq. (7.7) simplifies to,

$$\delta S = \int \sqrt{h}d^4x[(J^{\mu}h_{\mu\rho} + R_{\rho})\delta\tau^{\rho} - (\nabla_{\rho}J^{\rho})\chi] \tag{7.10}$$

For arbitrary χ, $\delta\tau^{\rho} = 0$ gives,

$$\nabla_{\rho}J^{\rho} = 0 \tag{7.11}$$

This is the equation for the conserved (matter) current in the theory. For arbitrary $\delta\tau^{\rho}$ and $\chi = 0$ we have from Eq. (7.10),

$$R_{\mu} = -J^{\rho}h_{\mu\rho} \tag{7.12}$$

This is the well known relation between the momentum and particle number currents in non-relativistic theories. Considering the variation of the action under diffeomorphisms one has,

$$0 = \delta S = \int \sqrt{h}d^4x[-\frac{1}{2}P_{\mu\nu}\pounds_{\xi}h^{\mu\nu} + Q^{\mu}\pounds_{\xi}\tau_{\mu} + J^{\mu}\pounds_{\xi}A_{\mu} + R_{\mu}\pounds_{\xi}\tau^{\mu}] \tag{7.13}$$

where $\pounds_{\xi}$ is the Lie derivative along some arbitrary vector field ξ^{μ}. After a bit more calculation Eq. (7.13) gives,

$$0 = \delta S = \int \sqrt{h}d^4x\ \xi^{\nu}[\nabla_{\mu}(-T^{\mu}{}_{\nu}) + 2J^{\mu}\nabla_{[\nu}A_{\mu]} + R_{\mu}\nabla_{\nu}\tau^{\mu}] \tag{7.14}$$

where

$$T^{\mu}{}_{\nu} = P_{\nu\rho}h^{\mu\rho} + Q^{\mu}\tau_{\nu} - R_{\nu}\tau^{\mu} \tag{7.15}$$

and

$$\nabla_{\mu}(T^{\mu}{}_{\nu}) = 2J^{\mu}\nabla_{[\nu}A_{\mu]} + R_{\mu}\nabla_{\nu}\tau^{\mu} = J^{\mu}K_{\nu\mu} + R_{\mu}\nabla_{\nu}\tau^{\mu} \tag{7.16}$$

To provide the constitutive relations we will now describe the physical currents of the theory in terms of fluid variables. For ideal fluids this involves the zeroth order derivative expansion. Since J^{μ} is some mass flow, we can write

$$J^{\mu}_{i} = \rho_i u^{\mu} \tag{7.17}$$

where ρ_i represents the conserved charge density. This choice is by no means exhaustive and in a general derivative expansion for dissipative fluids there exist more terms involving the spatial metric. At zeroth order in the derivative expansion we can also write Eq. (7.16) in the following form,

$$\nabla_\mu T^\mu{}_\nu = \rho h_{\nu\gamma} a^\gamma \tag{7.18}$$

We can now deduce the form of $T^\mu{}_\nu$ for ideal fluids. At this order the unknown coefficients $P_{\mu\nu}$, Q^μ and R_ν in Eq. (7.15) will not contain any derivatives of u^μ. Hence $T^\mu{}_\nu$ has the following general expression for ideal fluids

$$T^\mu{}_\nu = \alpha h_{\nu\rho} h^{\mu\rho} + \beta u^\mu \tau_\nu + \gamma h_{\nu\alpha} u^\alpha u^\mu \tag{7.19}$$

By performing the following contractions of $T^\mu{}_\nu$ with the expression of Eq. (7.19)

$$h^{\nu\alpha} h_{\alpha\mu} T^\mu{}_\nu = \alpha + \gamma u^\alpha u^\beta h_{\alpha\beta}\,, \quad \tau^\nu \tau_\mu T^\mu{}_\nu = \beta\,, \quad u^\nu \tau_\mu T^\mu{}_\nu = \beta + \gamma u^\alpha u^\beta h_{\alpha\beta}\,,$$

we see that Q^μ and R_μ can be interpreted as the energy and momentum currents respectively. This leads to the natural identification of $\beta = \epsilon + \frac{1}{2}\rho u^\alpha u^\beta h_{\alpha\beta}$ as the total energy of the fluid, $\gamma = -\rho$ to provide the momentum current and $\alpha = -P$. With these conventions for α , β and γ we have

$$T^\mu{}_\nu = (P + \epsilon + \frac{1}{2}\rho u^\alpha u^\beta h_{\alpha\beta}) u^\mu \tau_\nu - P\delta^\mu_\nu - \rho h_{\nu\alpha} u^\alpha u^\mu \tag{7.20}$$

The constitutive relation Eq. (7.20) for an ideal fluid on the NC backgroundis in agreement with the result of [51]. The stress tensor of the NC background, as in all theories with $z \neq 1$, satisfies a deformed trace relation $zT^0{}_0 + T^i{}_i = 0$. With the expression of Eq. (7.20) we find that this trace provides the following *Equation of state* when $z = 2$,

$$2\epsilon = (n-1)P \tag{7.21}$$

Note that this is a fully classical treatment. If quantum fluids were considered then this relation would follow from the 'dilatation Ward identity' associated with the Lifshitz symmetry [1].

Note that Eq. (7.20) represents the physical stress tensor and is not valid under Milne boosts. The velocity u^μ does not transform under the Milne transformation as u^μ is considered as a physical field. We recall that the set of Milne transformations that leave the (symmetric) connection invariant are

$$\begin{aligned}
\tau^\mu &\to \tau^\mu + h^{\mu\nu} k_\nu \\
h_{\mu\nu} &\to h_{\mu\nu} - 2\tau_{(\mu} k_{\nu)} + k^\alpha k^\beta h_{\alpha\beta} \tau_\mu \tau_\nu\,, \\
A_\mu &\to A_\mu + k_\mu - \frac{1}{2} k^\alpha k^\beta h_{\alpha\beta} \tau_\mu\,,
\end{aligned} \tag{7.22}$$

[1]This quantum relation is also known as 'z-deformed trace' [52] in the literature.

where k_μ is an arbitrary spatial vector, i.e. $\tau^\mu k_\mu = 0$. Under the Milne transformations Eq. (7.22), the variation of Eq. (7.20) is given by

$$\delta T^\mu_\nu = -\frac{\rho}{2} u^\mu \tau_\nu h^{\alpha\beta} k_\alpha k_\beta + \rho u^\mu k_\nu \tag{7.23}$$

There exist several ways in which Milne covariance can be assured. One approach involves redefining T^μ_ν such that

$$\begin{aligned}\tilde{T}^\mu{}_\nu &= T^\mu{}_\nu - \rho u^\mu A_\nu \\ &= (P + \epsilon + \frac{1}{2}\rho u^\alpha u^\beta h_{\alpha\beta}) u^\mu \tau_\nu - P\delta^\mu_\nu - \rho u^\mu u^\beta h_{\beta\nu} - \rho u^\mu A_\nu\end{aligned} \tag{7.24}$$

The stress tensor of Eq. (7.24) is invariant under Milne boosts and agrees with the expression of [59], where it was derived following the null reduction of a relativistic ideal fluid. A more systematic approach to ensure Milne invariance of all fluid relations to all orders involves the consideration of a Milne covariant formalism. This procedure was first described in [51]. Given a Milne invariant velocity u^μ, we define $u_\mu = h_{\mu\nu} u^\nu$ and $u^2 = u_\mu u^\mu$. We can now replace the Milne variant fields of the NC structure $(h_{\mu\nu}\,, \tau^\mu\,, A_\mu)$ with the new Milne invariant variables $(\tilde{h}_{\mu\nu}\,, u^\mu\,, \tilde{A}_\mu)$, where

$$\begin{aligned}\tilde{h}_{\mu\nu} &= h_{\mu\nu} - u_\mu \tau_\nu - u_\nu \tau_\mu + u^2 \tau_\mu \tau_\nu \\ \tilde{A}_\mu &= A_\mu + u_\mu - \frac{1}{2}\tau_\mu u^2\end{aligned} \tag{7.25}$$

In this way, beginning with any theory on the NC background, we can transform the variables into the new Milne invariant variables. This is particularly important in the case of the NC background with torsion, since the connection in that case is not simultaneously U(1) and Milne invariant. We will continue to work with the original set of variables of the NC background, as they allow for a clear relation to the scale covariant NC background to be considered next. In the resulting equations, we can always transform to the Milne invariant expressions using the transformations just described.

Another conservation equation we will be interested in involves the local entropy current. It follows from the second law of thermodynamics as a derived notion. The requirement that entropy should be non-decreasing during hydrodynamic evolution can be expressed in a covariant way in terms of an entropy current whose divergence is non-negative.

$$\nabla_\mu J^\mu_S \geq 0 \tag{7.26}$$

In Eq. (7.26) the equality holds for ideal fluids. The entropy current J^μ_S can be expressed as,

$$J^\mu_S = s' u^\mu \tag{7.27}$$

where 's'' is the entropy density of the fluid.

7.2 Fluids on the scale covariant Newton-Cartan background

In this subsection, we first introduce a manifestly Weyl-covariant formalism suited to the study of non-relativistic conformal incompressible fluids. An important feature of incompressible fluids is that the Euler equations are invariant under the scale transformation but not under the special conformal transformation [60]. Thus non-relativistic conformal incompressible fluids are only scale invariant, and the formulation we present here would be relevant in their description on curved backgrounds.

We assume that our system comprises of tensors $\tilde{Q}^{\alpha\cdots}_{\beta\ldots}$ which have a definite scale transformation, i.e. they obey $\tilde{Q}^{\alpha\cdots}_{\beta\ldots} = e^{ws}Q^{\alpha\cdots}_{\beta\ldots}$, where w is the scaling weight under scale transformations. Correspondingly, we also have the covariant derivative operator $\widetilde{\nabla}$ which satisfies

$$\widetilde{\nabla}_\mu V^\lambda_\rho = \nabla_\mu V^\lambda_\rho + (b_\mu\delta^\lambda_\nu + b_\nu\delta^\lambda_\mu - b_\sigma h^{\lambda\sigma}h_{\nu\mu})V^\nu_\rho - (b_\mu\delta^\nu_\rho + b_\rho\delta^\nu_\mu - b_\sigma h^{\nu\sigma}h_{\rho\mu})V^\lambda_\nu \tag{7.28}$$

where ∇_μ is the usual NC covariant derivative and b_μ is the scale gauge field of the previous chapter. The fluid velocity on the scale covariant NC background transforms as $\tilde{u}^\mu = e^{-zs}u^\mu$, where z is the dynamical exponent. Given our consideration of the NC background and our interest in the Schrödinger field in particular, we will consider the case where $z = 2$. However, we will also indicate the results which will follow for general z for many of the subsequent equations. Our analysis will be carried out in d spatial dimensions.

Using the above definitions, we can now write the general expression for $\widetilde{\nabla}_\mu\tilde{u}^\nu$

$$\widetilde{\nabla}_\mu\tilde{u}^\nu = e^{-zs}\left[(1-z)b_\mu u^\nu + \nabla_\mu u^\nu + \left(b_\alpha\delta^\nu_\mu - b_\sigma h^{\sigma\nu}h_{\mu\alpha}\right)u^\alpha\right] \tag{7.29}$$

With Eq. (7.29) and Eq. (7.5) we find that the expansion, acceleration, shear and vorticity have the following transformations,

$$\begin{aligned}
\tilde{\theta} &= \widetilde{\nabla}_\mu\tilde{u}^\mu = e^{-zs}\left[(d+2-z)b_\mu u^\mu + \theta\right]\\
\tilde{a^\nu} &= \tilde{u}^\mu\widetilde{\nabla}_\mu\tilde{u}^\nu = e^{-2zs}\left[(2-z)b_\mu u^\mu u^\nu + a^\nu - b_\sigma h^{\sigma\nu}u^2\right]\\
\widetilde{\sigma}^{\mu\nu} &= e^{-(2+z)s}\left[\sigma^{\mu\nu} + (1-z)b_\lambda h^{\lambda(\mu}u^{\nu)} + b_\lambda u^\lambda h^{\mu\nu}\right]\\
\widetilde{\omega}^{\mu\nu} &= e^{-(2+z)s}\left[\omega^{\mu\nu} + (1-z)b_\lambda h^{\lambda[\mu}u^{\nu]}\right]
\end{aligned} \tag{7.30}$$

where θ, a^ν, $\sigma^{\mu\nu}$ and $\omega^{\mu\nu}$ are defined in Eq. (7.5).

The above set of equations motivate the introduction of a conformally invariant covariant derivative '$\mathcal{D}$' such that for the tensor $\tilde{Q}^{\alpha\cdots}_{\beta\ldots}$ described above, the derivative will act on it as,

$$\mathcal{D}\tilde{Q}^{\alpha\ldots}_{\beta\ldots} = e^{ws}\mathcal{D}Q^{\alpha\ldots}_{\beta\ldots} \tag{7.31}$$

This leads to the following relation between $\mathcal{D}$ and $\tilde{\tilde{\nabla}}$

$$\mathcal{D}_\mu = \tilde{\tilde{\nabla}}_\mu - wb_\mu \tag{7.32}$$

Note that the above covariant derivative is metric compatible.

$$\mathcal{D}_\mu h^{\mu\nu} = 0,\ \mathcal{D}_\mu \tau_\mu = 0 \tag{7.33}$$

For relativistic conformal ideal fluids, the conformal acceleration ($u^\mu \mathcal{D}_\mu u^\alpha$) and conformal expansion ($\mathcal{D}_\mu u^\mu$) are assumed to vanish, leading to an expression for 'b_μ' in terms of the acceleration and expansion. We can identify a similar relation for the $z = 2$ non-relativistic case using the first two equations of Eq. (7.30). The requirements that $u^\mu \mathcal{D}_\mu u^\alpha = 0$ and $\mathcal{D}_\mu u^\mu = 0$ when $z = 2$ can easily be shown to lead to the following relation

$$b_\mu = -\frac{\theta}{d}\tau_\mu + \frac{a_\mu}{u^2} \tag{7.34}$$

As can be seen from Eq. (7.34) the conformally invariant derivative is useful in casting the variables and equations of non-relativistic fluid mechanics in a manifestly conformal language. These derivatives also define a curvature tensor through their commutator,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]V^\lambda = \tilde{R}^\lambda{}_{\sigma\mu\nu}V^\sigma - \omega F_{\mu\nu}V^\lambda \tag{7.35}$$

where $F_{\mu\nu} = \nabla_\mu b_\nu - \nabla_\nu b_\mu$, and $\tilde{R}^\lambda{}_{\sigma\mu\nu}$ is as given in Eq. (6.41). Note that should the usual symmetries of the Riemann tensor be assumed in Eq. (6.41), the field strength for the scale gauge field b_μ would necessarily vanish. This is in accordance with the present subsection where the usual symmetries follow through our choice of $b_\mu = \partial_\mu s$. While inconsequential for the case of ideal fluids, $F_{\mu\nu}$ does affect the derivative expansion and dissipative terms which result in non-ideal relativistic fluids [58].

Let us now use this derivative to describe the conservation equations of ideal fluids on the scale covariant NC background. For the stres tensor we consider Eq. (7.18) and find that

$$\mathcal{D}_\mu T^\mu_\nu = \rho a_\nu \tag{7.36}$$

provided T^μ_ν has weight '$d + z$' and satisfies $zT^0{}_0 + T^i{}_i = 0$. It thus follows from Eq. (7.20) that the conformal weights of 'P' and 'ϵ' are both '$d + z$', while the weight for 'ρ' is '$d+2-z$'. The acceleration 'a_ν' has the same weight as u^2 which is '$2z - 2$', with which 'ρa_ν' has the expected weight of '$d+z$'. These weights are the Lifshitz generalization of familiar results for relativistic fluids.

Likewise for any current $J^\mu_i = \rho_i u^\mu$ we find that

$$\mathcal{D}_\mu J^\mu = 0 \tag{7.37}$$

when J_i^μ has the scaling weight of '$(d+2)$'. Thus all current densities ρ_i have the weight '$d+2-z$', including the entropy current relevant to the thermodynamics of the fluid.

We assume that our fluid is in local thermodynamic equilibrium in the neighbourhood of any point of spacetime. Let us denote the entropy current density as 's'', the temperature as 'T' and the chemical potentials as 'μ^i'. The first law of thermodynamics for this system can be written in terms of the conformally invariant derivative as,

$$T u^\lambda \mathcal{D}_\lambda s' = \frac{(n-1)}{2} u^\lambda \mathcal{D}_\lambda P - \mu^i u^\lambda \mathcal{D}_\lambda \rho_i \tag{7.38}$$

It can be noted that the weight of 'T' is 'z' while that of 'μ^i' be '$2z-2$'. For ideal fluids, it follows from Eq. (7.17) and Eq. (7.37) that $\mu^i u^\lambda \mathcal{D}_\lambda \rho_i = 0$. Likewise, Eq. (7.18) shows that $u^\nu \mathcal{D}_\mu T^\mu_\nu = 0$, which establishes that $u^\lambda \mathcal{D}_\lambda P = 0$ on substituting Eq. (7.20). Thus the entropy density for an ideal fluid on the scale covariant NC background satisfies the following relation

$$T u^\alpha \mathcal{D}_\alpha s' = 0 \tag{7.39}$$

This implies that the ideal incompressible fluids on the scale covariant NC background satisfies the local second law of thermodynamics i.e. the motions of the fluid conserve the entropy of the system and no heat flows in or out of the fluid during its motion.

Having considered scale invariant ideal fluids in this subsection, we already noted some key differences with the relations that result from relativistic backgrounds. It will be essential to further consider the description of fluids at higher orders in the derivative expansion. This can be carried out using the derivative provided in Eq. (7.32) and the field strength constructed from it Eq. (7.35), along with the Riemann tensor relations Eq. (6.41) - Eq. (6.44). The main complication involves the Riemann tensors, which despite admitting a Weyl tensor description, require many more constraints than the usual relativistic construction. Further, the Riemann tensor relations were determined in the absence of torsion. As noted in Chapter 6, the inclusion of torsion is particularly warranted in the case of the scale covariant NC background.

7.3 Contributions of scale symmetry to the Hall Effect

In this section, we will be interested in the consequences of non-relativistic anisotropic scale symmetry in describing Hall fluids. We will follow the procedure described in [43] where the Hall viscosity and the Wen-Zee term are derived using an effective hydrodynamic theory. The Hall viscosity results from the Berry phase term in the effective action [43]. More specifically, it is the response to spatial stress in the corresponding term of the stress energy tensor. The effective field theory consists of the Schrödinger field minimally coupled to a background electromagnetic field $\mathcal{A}_\mu$, and a "dynamical" statistical field a_μ. The inclusion of the Chern Simons term involving the field a_μ follows from the need to study perturbations about a mean field of a strongly coupled anyonic system. The statistical field term in effect fixes the statistics of the system to be either bosonic or fermionic and enables the study of responses to the system. After the perturbation has been taken into account, one can then integrate out this field to have the effective field theory description of the Hall fluid. In this context, the field Φ represents either a composite boson or a composite fermion, and since we are interested in the consequences of curved backgrounds on the system, we will investigate the former. We can use the result of Section 4.3 to express the Chern Simons Landau Ginzburg (CSLG) effective action of the Quantum Hall effect [42] in the following way,

$$\begin{aligned} S = \int dt d^2x \sqrt{h} \Big[& \frac{i}{2}\tau^\mu \left(\Phi(x) D_\mu \Phi(x)^* - \Phi^*(x) D_\mu \Phi(x)\right) - \frac{1}{2m} h^{\mu\nu} (D_\mu \Phi(x))^* (D_\nu \Phi(x)) \\ & - V(\Phi^*\Phi) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi g} a_\mu \nabla_\nu a_\lambda \Big] \end{aligned} \tag{7.40}$$

where $\varepsilon^{\mu\nu\lambda}$ is the Levi Civita tensor, and the covariant derivative on the curved background 'D_μ' is,

$$\begin{aligned} D_\mu &= \partial_\mu + ie\mathcal{A}_\mu + ia_\mu + igB_\mu + ig'C_\mu \\ &= \partial_\mu + i\alpha_\mu + ia_\mu \,, \end{aligned} \tag{7.41}$$

In Eq. (7.41) '$\mathcal{A}_\mu$' is the external electromagnetic field, 'a_μ' is the statistical gauge field, 'B_μ' was introduced at the time of localization of the Galilean symmetry and similarly 'C_μ' for the scale transformation in Eq. (3.18). Since we will integrate out the statistical field a_μ before our final result, it will be useful to write the covariant derivative as in the second equality of Eq. (7.41). The hydrodynamic version of Eq. (7.40) is derived by expressing the complex field Φ in polar variables [43, 61],

$$\Phi = \sqrt{\rho} e^{i\theta} \tag{7.42}$$

where ρ is the matter density, $\rho = \Phi^*\Phi$. The transformation Eq. (7.42) leads to the following action,

$$\begin{aligned} S = \int dt d^2x \sqrt{h} [& \rho\tau^\mu \left(\partial_\mu\theta + \alpha_\mu + a_\mu\right) - \frac{\rho}{2m} h^{\mu\nu} \left(\partial_\mu\theta + \alpha_\mu + a_\mu\right)\left(\partial_\nu\theta + \alpha_\nu + a_\nu\right) \\ & - \frac{1}{8m\rho} h^{\mu\nu} \partial_\mu\rho \partial_\nu\rho - V(\rho) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi g} a_\mu \nabla_\nu a_\lambda] \end{aligned} \tag{7.43}$$

The response of the FQH state follows from the variations of the fields

$$\begin{aligned} \rho &\to \bar{\rho} + \delta\rho \\ \mathcal{A}_\mu &\to \bar{\mathcal{A}}_\mu + \delta\mathcal{A}_\mu \\ a_\mu &\to \bar{a}_\mu + \delta a_\mu \end{aligned} \tag{7.44}$$

where the barred values represent the mean field values. The FQH state of the electron corresponds to the superfluid state of the boson Φ, where $\bar{\mathcal{A}}_\mu$ is completely cancelled by $\bar{a}_\mu$. Further, for the Hall fluid the average density, $\bar{\rho}$, is related to the fields $\bar{\mathcal{A}}_\mu$ as,

$$\bar{\rho} = \frac{1}{4\pi g}\varepsilon^{0ij}\nabla_i\bar{\mathcal{A}}_j = -\frac{1}{4\pi g}\varepsilon^{0ij}\nabla_i\bar{a}_j \tag{7.45}$$

where the filling fraction in Eq. (7.45) is written in terms of the intrinsic orbital spin 'g' through the relation $\nu = \frac{1}{2g}$. With these considerations at hand, the effective action Eq. (7.43) modifies to the following one, where we will retain terms that are at most quadratic in variations and derivatives.

$$\begin{aligned}\mathcal{L} = \sqrt{h}\Big[&\tau^\mu(\partial_\mu\theta + \delta\alpha_\mu)\bar{\rho} + \tau^\mu(\partial_\mu\theta + \delta\alpha_\mu + \delta a_\mu)\delta\rho \\ &- \frac{\bar{\rho}h^{\mu\nu}}{2m}(\partial_\mu\theta + \delta\alpha_\mu + \delta a_\mu)(\partial_\nu\theta + \delta\alpha_\nu + \delta a_\nu) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi g}\delta a_\mu\nabla_\nu\delta a_\lambda - V(\bar{\rho})\Big]\end{aligned} \tag{7.46}$$

We can now introduce a field j^μ through a Hubbard-Stratonovich transformation on the kinetic term of the action in Eq. (7.46) to rewrite the action as,

$$\begin{aligned}\mathcal{L} =& \sqrt{h}\Big[\tau^\mu(\partial_\mu\theta + \delta\alpha_\mu)\bar{\rho} + \tau^\mu(\partial_\mu\theta + \delta\alpha_\mu + \delta a_\mu)\delta\rho - (\partial_\mu\theta + \delta\alpha_\mu + \delta a_\mu)h^{\mu\nu}j_\nu \\ &+ \frac{m}{2\bar{\rho}}j_\mu h^{\mu\nu}j_\nu - V(\bar{\rho}) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi g}\delta a_\mu\nabla_\nu\delta a_\lambda\Big],\end{aligned} \tag{7.47}$$

In the absence of the vortex excitation, we can integrate out the phase variable θ in Eq. (7.47) to find the following conservation equation,

$$\partial_\mu(\sqrt{h}J^\mu) = \sqrt{h}\nabla_\mu J^\mu = 0 \tag{7.48}$$

where we have defined $J^\mu = \delta\rho\tau^\mu - j_\nu h^{\nu\mu}$. Given Eq. (7.48) holds, we can further express it as,

$$J^\mu = \varepsilon^{\mu\nu\lambda}\frac{1}{2\pi}\nabla_\nu f_\lambda\,, \tag{7.49}$$

where f_λ are the new hydrodynamic gauge variables. Clearly, J^μ remains invariant under $U(1)$ transformations of the field f_λ. By substituting this expression for J^μ back in Eq. (7.47), we find,

$$\mathcal{L} = \sqrt{h}\Big[\bar{\rho}\tau^\mu\delta\alpha_\mu + \varepsilon^{\mu\nu\lambda}\frac{1}{2\pi}\nabla_\nu f_\lambda(\delta\alpha_\mu + \delta a_\mu) + \frac{m}{2\bar{\rho}}j_\mu h^{\mu\nu}j_\nu + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi g}\delta a_\mu\nabla_\nu\delta a_\lambda - V(\bar{\rho})\Big] \tag{7.50}$$

Integrating out δa_μ and using the expression of $\delta\alpha_\mu$ from Eq. (7.41) we obtain this effective theory for the Hall state on the scale invariant Newton-Cartan background upto the leading order in gauge fields,

$$\mathcal{L} =\sqrt{h}\left[(g\tau^\mu B_\mu\bar{\rho} + g'\tau^\mu C_\mu\bar{\rho}) + (\frac{g}{2\pi}\varepsilon^{\mu\nu\lambda}B_\mu\partial_\nu f_\lambda + \frac{g'}{2\pi}\varepsilon^{\mu\nu\lambda}C_\mu\partial_\nu f_\lambda) \right.$$
$$\left. +e\tau^\mu\delta\mathcal{A}_\mu\bar{\rho} + \frac{e}{2\pi}\varepsilon^{\mu\nu\lambda}\delta\mathcal{A}_\mu\partial_\nu f_\lambda - \frac{g}{2\pi}\varepsilon^{\mu\nu\lambda}f_\mu\partial_\nu f_\lambda + \cdots\right] \quad (7.51)$$

The first parenthesis in Eq. (7.51) represents the Berry phase terms and the terms in the second parenthesis are the Wen-Zee terms. The terms with coefficient 'g' have arisen due to the symmetries of Newton-Cartan background. The terms involving 'g'' are the contributions of additional scale symmetry. Our aim is to study the response of the effective action Eq. (7.51) to the time dependent variation of spatial metric. This response receives contributions only from those terms which are quadratic in variations of the spatial metric under the presence of a constant magnetic field ($\bar{\rho} = \text{const.}$). Hence only the Berry phase terms will be relevant to study the contribution to the Hall viscosity through the stress tensor. The Wen-Zee terms will change the flux due to the curved background in a time independent manner.

We will consider the time dependent variations of the spatial metric and its inverse about flat space, which we will label as $\delta h_{\mu\nu}(t)$ and $\delta h^{\mu\nu}(t)$ respectively. In doing so with Eq. (7.51), we end up with the following contribution which is quadratic in variations,

$$L_2 = \frac{1}{8}g\bar{\rho}\,\epsilon_{ab}\delta^{a\mu}\delta^b_\nu\left(\delta h_{\mu\rho}\delta\dot{h}^{\rho\nu}\right) + \frac{1}{4}g'\bar{\rho}\delta h_{\mu\rho}\delta\dot{h}^{\mu\rho} + \cdots \quad (7.52)$$

where the overdot implies the time derivative and '$\cdots$' denotes those terms other than quadratic order, which have been neglected in L_2. Using Eq. (7.52), we find the following correction to the stress tensor,

$$T^\mu_\nu = \frac{\eta_H}{2}\left(\frac{1}{2}\epsilon_{ab}\delta^{a\mu}\delta^b_\sigma\delta h_{\lambda\nu}\delta\dot{h}^{\lambda\sigma} - \frac{1}{2}\epsilon_{ab}\delta^{a\sigma}\delta h_{\sigma\lambda}\delta^b_\nu\delta\dot{h}^{\lambda\mu}\right) + \frac{\theta_H}{2}\partial_t\left(\delta h^{\mu\sigma}\delta h_{\sigma\nu}\right) \quad (7.53)$$

where we have denoted $\frac{g\bar{\rho}}{2} = \eta_H$ and $\frac{g'\bar{\rho}}{2} = \theta_H$. In deriving Eq. (7.53) we made use of the fact that $\epsilon_{0b} = 0$. The term in the parenthesis of Eq. (7.53) is the Hall component of the viscosity tensor, which justifies our notation for η_H. The second term follows due to our consideration of the scale covariant NC background. In involving the time derivative of the spatial metric variations this additional term rescales the Hall fluid. We note that the spatial metric variations must also be related to corresponding temporal variations of τ^μ, so as to satisfy Eq. (5.4). As such, this term may also be viewed as an expansion of the Hall droplet which results in order to preserve the scale invariance of the Newton-Cartan background.

Our analysis here was entirely classical. One can expect that quantum effects, especially one-loop effects, will be relevant in the description of Schrödinger field theories and fluids on the NC background. The next chapter addresses this topic through the derivation of the trace and diffeomorphism anomalies of the Schrödinger field on the NC background.

Chapter 8

Newton-Cartan gravitational anomalies of the Schrödinger field

Anomalies are one loop effects which arise in the context of quantum fields coupled to external gauge fields or gravitational backgrounds. They represent the failure of classical conservation laws to hold at the quantum level. Classical relativistic systems admit a stress-energy tensor, which is symmetric, traceless and conserved. But in consideration of quantum fields the trace anomaly arises when the quantum stress-energy tensor is not traceless and its failure to be conserved results in the diffeomorphism anomaly. These anomalies have important consequences in black holes physics and cosmology [62–68], as well as in the computation of transport coefficients and response functions of condensed matter systems [69–76]. In particular, the trace anomaly is known to be relevant in describing the RG flow of quantum field theories [77–81] and in particular has led to a proof of the a-theorem in CFTs [82]. NC gravitational anomalies will also be relevant for certain systems with boundaries. As the AdS/CFT correspondence is expected to hold in the NR limit, the bulk anomalies in $2+1$ dimensions are expected to impose certain constraints on the nature of the dual field theory at the boundary [83–86].

Gravitational anomalies can be calculated using several approaches. In the following we will briefly discuss some of the most commonly used ones in the case of relativistic backgrounds.

Path Integral derivation with Pauli-Villars regularization: The anomaly results due to perturbations about a flat background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Minkowski background and $h_{\mu\nu}$ denotes the perturbation. This is usually performed within the Pauli-Villars scheme, which is known to result in the derivation of consistent gravitational anomalies.

Heat kernel approach: Gravitational anomalies can also be derived from the trace of the heat kernel [87, 88]. Let us consider quantum fields with a mode expansion

ϕ_n, which satisfy $\hat{L}\phi_n = \lambda_n \phi_n$. The use of elliptic operators ensures that this spectra is bounded. Then the heat kernel $K(x,y;s)$ is given by

$$K(x,y;s) = \langle x|e^{-s\hat{L}}|y\rangle \tag{8.1}$$

where s is the coefficient for the operator $\hat{L}$ which ensures that the exponent in Eq. (8.1) is dimensionless. The heat kernel satisfies the following heat equation

$$(\partial_s + \hat{L})K(x,y;s) = 0 \tag{8.2}$$

For the Laplacian operator in flat space this kernel is known exactly. In curved space, we can approximate the full expression through a perturbation in s. In this way the heat trace can be expressed as,

$$K(x,x;s) = \frac{1}{s^{\frac{d}{2}+1}}\left(a_0(\hat{L}) + a_2(\hat{L})s + a_4(\hat{L})s^2 + \cdots\right) \tag{8.3}$$

where 'd' represents the spatial dimensions and a_i are the anomaly coefficients. Depending on the degree of the operator $\hat{L}$ and the dimensions of the spaceitme, one of these coefficients will represent the anomaly. It is also implicitly assumed in Eq. (8.3) that $\hat{L}$ has an even mass dimension due to which all the coefficients a_i are even.

Fujikawa's approach: Anomalies can also be seen as the failure of the measure of the path integral to remain invariant under the given symmetry transformation. The functional trace of the Jacobian for gravitational anomalies requires the choice of regulator and basis. One such choice involves the plane wave approach of [89,90]. This approach leads to the correct result for the relativistic trace, chiral and diffeomorphism anomalies. The regulator was introduced in [91], which was further shown to be equivalent to Pauli-Villars regularization in [92].

The symmetric stress tensor of the Schrödinger field on the Newton-Cartan background can possess three possible gravitational anomalies. These are the diffeomorphism, trace and the gravitational U(1) anomalies. The gauge field A_μ[1] is contained in the NC connection due to which the U(1) anomaly is also a gravitational anomaly. The path integral and heat kernel approaches described above are effective in the relativistic case since calculations only involve the pertubed metric $h_{\mu\nu}$. In constrast, the NC background involves perturbations of the fields $h^{\mu\nu}\,,\tau_\mu\,,\tau^\mu$ and A_μ, leading to a substantially more involved calculation. Further, NC anomalies derived thus far in the literature all concern the trace anomaly, where variations of A_μ were not considered. Beginning with [93], the trace anomaly was described as those terms in the general Weyl variation which satisfy the Wess-Zumino consistency condition. In [94], following the null background construction of [39], the anomaly was shown to be present in the same number of dimensions as relativistic theories. In [95], the trace anomaly was demonstrated to arise in odd dimensions, following the embedding of the NC

[1]This field is related to the field B^{a0}_μ considered in Chapter 5.

background in a relativistic background of one dimension higher [27]. In this chapter, we will review the derivation of the NC trace and diffeomorphism anomalies following Fujikawa's approach, where all variations of the Newton-Cartan fields are taken into account.

8.1 Fujikawa's approach and Regulators

Fujikawa's approach :

We will consider the action $S[\Psi, \mathcal{G}]$ which is invariant under certain linear transformations of the fields

$$\delta S = \frac{\delta S}{\delta \Psi}\delta\Psi + \frac{\delta S}{\delta \mathcal{G}}\delta\mathcal{G} = 0\,, \tag{8.4}$$

where Ψ are the matter fields, $\mathcal{G} = \{h^{\mu\nu}, \tau^{\mu}, \tau_{\mu}, A_{\mu}\}$ are the background (gravitational) fields for the NC background and $\frac{\delta S}{\delta \mathcal{G}}$ is the densitized energy-momentum tensor. For the on-shell equations of motion of Ψ, the first term on the right hand side of Eq. (8.4) vanishes, while the second term provides the classical conservation equation of the energy-momentum tensor

$$\frac{\delta S}{\delta \mathcal{G}}\delta\mathcal{G} = 0\,. \tag{8.5}$$

Eq. (8.5) can represent either the Weyl or diffeomorphism transformations in the context of this chapter. The quantum theory is described by the path integral

$$Z = \int \mathcal{D}\Psi e^{iS[\Psi,\mathcal{G}]}\,, \tag{8.6}$$

The path integral is invariant under a given symmetry transformation of Ψ provided

$$\int \mathcal{D}\Psi' e^{iS[\Psi',\mathcal{G}]} = \int \mathcal{D}\Psi e^{iS[\Psi,\mathcal{G}]}\,. \tag{8.7}$$

We will be interested in infinitesimal transformations, under which the left hand side of Eq. (8.7) can be expressed as

$$\int \mathcal{D}\Psi' e^{iS[\Psi',\mathcal{G}]} = \int \mathcal{D}\Psi\ \mathcal{I} e^{i\left(S[\Psi,\mathcal{G}] + \frac{\delta S}{\delta\Psi}\delta\Psi\right)}\,, \tag{8.8}$$

where $\mathcal{I}$ refers to the functional Jacobian in going from Ψ to Ψ'. The effect of infinitessimal changes to the Jacobian and the action will provide the anomalous Ward identity. The infinitesimal change in the action is given by

$$\begin{aligned} S[\Psi',\mathcal{G}] &= S[\Psi,\mathcal{G}] + \frac{\delta S}{\delta\Psi}\delta\Psi \\ &= S[\Psi,\mathcal{G}] - \frac{\delta S}{\delta\mathcal{G}}\delta\mathcal{G} \end{aligned} \tag{8.9}$$

where in going from the first to the second line of Eq. (8.9) we made use of Eq. (8.4). We also have the unitary transformation of the field Ψ, which can be written as

$$\Psi' = U\Psi = e^{iJ}\Psi\,, \tag{8.10}$$

where J is the Jacobian of the transformation. Using Eq. (8.9) and Eq. (8.10) in Eq. (8.7) now leads to the anomalous Ward identity

$$\left\langle \frac{\delta S}{\delta \mathcal{G}} \delta \mathcal{G} \right\rangle_\Psi = \langle \mathrm{Tr} J \rangle_\Psi \quad , \tag{8.11}$$

where $\langle \cdots \rangle_\Psi$ denotes the path integral average with respect to the variable Ψ. Thus the classical conservation equation is violated and results in an anomaly which is given by the functional trace of the Jacobian. This trace is ill-defined due to the presence of $\delta(0)$ and requires regularization. As first demonstrated by Fujikawa [96], one can regulate using a positive definite operator $\mathcal{R}$ in the following way

$$\mathrm{An} = \lim_{M\to\infty} \mathrm{Tr}\left[J e^{\frac{\mathcal{R}}{M^2}}\right] = \lim_{M\to\infty} \int d^n x \int d^n y J(x,y) e^{\frac{\mathcal{R}(x)}{M^2}} \delta^n(x-y)\,, \tag{8.12}$$

where the mode expansion for the functional trace in the last equality has been made for a scalar field. M^2 represents a mass parameter meant to make the exponent in Eq. (8.12) dimensionless. Its purpose in the regulated trace is to eliminate the UV divergences, thereby leading to a finite result for the anomaly in the limit $M \to \infty$. In Eq. (8.12), the true anomaly comprise only those terms for which a counterterm in the action cannot be provided.

Regulators: Pauli-Villars (PV) regularization can be used to infer the Jacobian (J) and Regulator ($\mathcal{R}$) of Eq. (8.12) [90]. Let us consider the following action involving a collection of quantum fields Ψ

$$\mathcal{L}_\Psi = \frac{1}{2}\Psi^T \mathbf{T} \mathcal{Q} \Psi\,, \tag{8.13}$$

where we assume that $\mathcal{Q}$ is any symmetric operator of mass dimension 2. The superscript T denotes transposition, while $\mathbf{T}$ is a symmetric matrix which in general depends on the background fields. Eq. (8.13) is invariant under a certain infinitesimal symmetry transformation which we denote as

$$\delta_K \Psi = K\Psi\,. \tag{8.14}$$

where K is the generator of the corresponding transformation.

In order to regularize the action we introduce the PV fields χ. These are massive fields with the same statistics as Ψ, but with a different path integral definition to introduce a minus sign in one-loop graphs. Thus the Lagrangian is

$$\begin{aligned} \mathcal{L}_{PV} &= \mathcal{L}_\chi + \mathcal{L}_M \\ &= \frac{1}{2}\chi^T \mathbf{T} \mathcal{Q} \chi + \frac{1}{2} M^2 \chi^T \mathbf{T} \chi\,, \end{aligned} \tag{8.15}$$

where M^2 in the second term is due to $\mathcal{Q}$ in Eq. (8.13) being a mass dimension 2 operator. The path integral for χ is defined as

$$\int \mathcal{D}\chi e^{i\chi^T A\chi} = (\det A)^{\frac{1}{2}} \tag{8.16}$$

The invariance of Eq. (8.13) is now extended to the massless part of the PV action [2]

$$\delta_K \chi = K\chi \,, \tag{8.17}$$

such that the violation of symmetries, if any, can only arise due to the mass term. Under the transformation Eq. (8.17) the mass term of the PV Lagrangian becomes

$$\delta_K \mathcal{L}_M = \delta_K \mathcal{L}_{PV} = \frac{1}{2} M^2 \chi^T \left(\mathbf{T}K + K^T\mathbf{T} + \delta\mathbf{T} \right) \chi \,. \tag{8.18}$$

Eq. (8.18) can now be used to compute the anomaly due to the PV regulated path integral

$$\begin{aligned} \mathrm{An}_K &= - \lim_{M\to\infty} \mathrm{Tr} \left[\frac{1}{2} M^2 \left(\mathbf{T}K + K^T\mathbf{T} + \delta\mathbf{T} \right) \left(\mathbf{T}M^2 + \mathbf{T}\mathcal{Q} \right)^{-1} \right] \\ &= - \lim_{M\to\infty} \mathrm{Tr} \left[\left(K + \frac{1}{2}\mathbf{T}^{-1}\delta\mathbf{T} \right) \left(1 + \frac{\mathcal{Q}}{M^2} \right)^{-1} \right] \,, \end{aligned} \tag{8.19}$$

where we could replace $K^T\mathbf{T}$ with $\mathbf{T}K$ since $\mathbf{T}$ and $\mathbf{T}\mathcal{Q}$ are symmetric. From Eq. (8.12) and Eq. (8.19), we can identify the Jacobian and the regulator to be used in Fujikawa's approach as

$$J = K + \frac{1}{2}\mathbf{T}^{-1}\delta\mathbf{T} \,, \qquad \mathcal{R} = \mathcal{Q} \tag{8.20}$$

8.2 Fujikawa regulators for non-relativistic field theories

While the comparison of PV regularization with that of the regulated trace in Fujikawa's approach has led to Eq. (8.20), one aspect of the calculation in the PV scheme is not faithfully represented for non-relativistic systems. This concerns the domain of integration of ω in a non-relativistic one-loop calculation. Specifically, we will now argue that the correct regulated trace to be used in the Fujikawa approach for non-relativistic theories should be

$$\lim_{M\to\infty} \mathrm{Tr} J = \lim_{M\to\infty} \int_0^\infty \frac{d\omega}{2\pi} \int_{-\infty}^\infty \frac{d^2k}{(2\pi)^2} e^{-i\omega t} e^{ikx} \left[J(x) e^{\frac{\mathcal{R}}{M^2}} \right] e^{i\omega t} e^{-ikx} \,. \tag{8.21}$$

[2]Strictly speaking, we can have $\delta_{K'}\chi = K'\chi$, but in this case K' must be such that the Jacobians of the fields χ and Ψ cancel out. Here for simplicity, we have referred to K' as K.

We recall that while one-loop effects in relativistic field theories involve pair creation and annhilation processes, vaccuum polarization effects, charge renormalization and mass renormalization, such processes are absent at one-loop for non-relativistic field theories [37, 97]. The reason for this is that we can either have the forward time or the retarded time propagator. To understand what happens in the non-relativistic case let us first consider the Schrödinger field in $2+1$ dimensions. Its mode expansion in terms of non-relativistic plane waves is given by

$$\begin{aligned}\Phi(x) &\sim e^{i\omega t - ikx} \\ \Phi^*(x) &\sim e^{-i\omega t + ikx}\end{aligned} \tag{8.22}$$

Noting Eq. (8.21), we now want to determine what should be the range of the ω integral due to the action of $\mathcal{R}$ on Φ. While we do not have access to the full Schrödinger propagator on curved backgrounds, it will suffice to consider the flat space operator to determine the nature of the ω integral. Taking $\mathcal{R} = i\partial_t + \frac{\nabla^2}{2}$, the propagator $G(x,t)$ satisfies

$$\left(i\partial_t + \frac{\nabla_x^2}{2}\right) G(x, x'; t, t') = \delta(t-t')\delta^2(x-x')\,, \tag{8.23}$$

With $x' = 0$ and $t' = 0$ for simplicity, we can use the Fourier transform to express the propagator as the following integral

$$G(x;t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{e^{i\omega t - ikx}}{\omega + \frac{k^2}{2}} \tag{8.24}$$

This integral can be evaluated by choosing a pole *either* in the upper half plane ($\omega \geq 0$) or the lower half plane ($\omega \leq 0$). This freedom allows us to choose either the forward or retarded propagator. Given Eq. (8.24) and the usual choice of the forward propagator for particles, this requires choosing the pole in the upper half plane

$$G(x;t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{e^{i\omega t - ikx}}{\omega + \frac{k^2}{2} - i\epsilon} \tag{8.25}$$

We can now readily integrate to find

$$G(x;t) = -\frac{\Theta(t)}{t} e^{-\frac{ix^2}{2t}} \tag{8.26}$$

The above calculation is what is involved in Eq. (8.19), but is not explicitly considered in Eq. (8.12). We will always consider the forward propagator for particles. Thus we could have performed the integration over ω in Eq. (8.25) from 0 to ∞ without affecting the result. As the Fujikawa approach is meant to convey a one-loop calculation with this propagator for particles, we will perform our calculation in Fujikawa's approach with the regulator provided in Eq. (8.21).

8.3 The Schrödinger field on the Newton-Cartan background

The Schrödinger field on the NC background in $2+1$ dimensions [7,8,26] can be written as[3],

$$S = \int dtd^2x\sqrt{h}\mathcal{L}$$
$$= \int dtd^2x\sqrt{h}\left[im\left(\Phi^*\tau^\mu\mathcal{D}_\mu\Phi - \Phi\tau^\mu\bar{\mathcal{D}}_\mu\Phi^*\right) - h^{\mu\nu}\mathcal{D}_\mu\Phi\bar{\mathcal{D}}_\nu\Phi^*\right] , \qquad (8.27)$$

where $\mathcal{D}_\mu = \nabla_\mu - imA_\mu$, $\bar{\mathcal{D}}_\mu = \nabla_\mu + imA_\mu$ and ∇_μ represents the usual covariant derivative of the spacetime. The covariant measure for the action is given by $\sqrt{h_{\mu\nu}+\tau_\mu\tau_\nu} = \tau^\mu\tau_\mu\sqrt{h} = \sqrt{h}$, which follows from Eq. (5.3) and Eq. (5.4). The gauge field A_μ is a mass generating field which provides particle number conservation on the NC background. It is also the same field which appears in the NC connection and is therefore on the same footing as all other gravitational fields. In addition, the action (Eq. (8.27)) is known to be invariant under Milne boosts [7]. It will be useful to define the Milne invariant quantities

$$v^\mu = \tau^\mu - h^{\mu\nu}A_\nu = \tau^\mu - A^\mu$$
$$\psi = \tau^\mu A_\mu - \frac{1}{2}h^{\mu\nu}A_\mu A_\nu \qquad (8.28)$$

We also define $\partial^\mu = h^{\mu\nu}\partial_\nu$. Note that in Eq. (8.27), m is merely a passive parameter with no mass dimension [97]. By dimensional analysis we see that Φ, Φ^*, $h^{\mu\nu}\partial_\mu$ and $h^{\mu\nu}A_\mu$ each have mass dimension 1, while $\tau^\mu A_\mu$ and $\tau^\mu\partial_\mu$ each have mass dimension 2. Since we are interested in understanding the symmetries of Eq. (8.27), let us first consider its total variation

$$\delta S = \int dtd^2x\sqrt{h}\left[-P_{\mu\nu}\delta h^{\mu\nu} + R_\mu\delta\tau^\mu - J^\mu\delta A_\mu + \delta\Phi^*\mathcal{D}\Phi + \delta\Phi\bar{\mathcal{D}}\Phi^*\right] , \qquad (8.29)$$

where we have defined

$$P_{\mu\nu} = \frac{1}{2}h_{\mu\nu}\mathcal{L} + \mathcal{D}_\mu\Phi\bar{\mathcal{D}}_\nu\Phi^*$$
$$R_\mu = im\left(\Phi^*\mathcal{D}_\mu\Phi - \Phi\bar{\mathcal{D}}_\mu\Phi^*\right)$$
$$J^\mu = -2m^2\Phi\Phi^*v^\mu + im\left(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*\right)$$
$$\mathcal{D}\Phi = \left(2im\tau^\mu\mathcal{D}_\mu + im\nabla_\mu\tau^\mu + h^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\right)\Phi$$
$$\bar{\mathcal{D}}\Phi^* = \left(-2im\tau^\mu\mathcal{D}_\mu - im\nabla_\mu\tau^\mu + h^{\mu\nu}\bar{\mathcal{D}}_\mu\bar{\mathcal{D}}_\nu\right)\Phi^* \qquad (8.30)$$

[3]The action here differs from that of the previous section by a factor of $2m$. We are free to perform this rescaling as m is a passive parameter in NRFTs and do not involve corrections in loop processes (no mass renormalization; particle number is conserved,etc.)

We note again that variations of $\delta h_{\mu\nu}$ are not independent of $\delta\tau^\mu$ and hence do not appear in Eq. (8.29). Let us now consider the variations to be the Lie derivative with respect to some arbitrary vector field ξ^μ, i.e. $\delta_\xi = \pounds_\xi$. It is straightforward to demonstrate that $\delta_\xi S = 0$ and hence Eq. (8.27) is invariant under diffeomorphisms.

We further consider the on-shell symmetries of the action

$$\begin{aligned} 0 = \delta_\xi S &= \int dt d^2x \sqrt{h}[-P_{\mu\nu}\pounds_\xi h^{\mu\nu} + R_\mu \pounds_\xi \tau^\mu - J^\mu \pounds_\xi A_\mu] \\ &= \int dt d^2x 2\sqrt{h}\xi^\nu[-\nabla_\mu T^\mu{}_\nu - J^\mu \nabla_{[\nu}A_{\mu]} + \frac{1}{2}R_\mu \nabla_\nu \tau^\mu] \end{aligned} \tag{8.31}$$

Here $T^\mu{}_\nu$ is the stress tensor of the Schrödinger field on the NC background, which is defined as

$$T^\mu{}_\nu = P_{(\nu\sigma)}h^{\sigma\mu} - \frac{1}{2}R_\nu \tau^\mu\,. \tag{8.32}$$

Thus Eq. (8.27) remains invariant under on-shell diffeomorphisms provided the stress tensor satisfies

$$\nabla_\mu T^\mu{}_\nu + J^\mu \nabla_{[\nu}A_{\mu]} - \frac{1}{2}R_\mu \nabla_\nu \tau^\mu = 0\,. \tag{8.33}$$

The action Eq. (8.27) is not Weyl invariant and cannot be used to investigate the Weyl anomaly. However, in $2+1$ dimensions we can construct a Weyl-invariant action from (Eq. (8.27)) by replacing the scalar fields with scalar densities. This trick is known to work for relativistic scalar fields in $1+1$ dimensions, where the densitized fields are known as Fujikawa variables. We will now demonstrate that this substitution also works for (Eq. (8.27)).

By substituting $\Phi = \widetilde{\Phi}h^{-\frac{1}{4}}$ and $\Phi^* = \widetilde{\Phi}^* h^{-\frac{1}{4}}$ in (Eq. (8.27)), we have

$$\begin{aligned} \widetilde{S} &= \int dt d^2x \sqrt{h}\widetilde{\mathcal{L}} \\ &= \int dt d^2x \sqrt{h}\left[imh^{-\frac{1}{4}}\left(\widetilde{\Phi}^*\tau^\mu \mathcal{D}_\mu(\widetilde{\Phi}h^{-\frac{1}{4}}) - \widetilde{\Phi}\tau^\mu \bar{\mathcal{D}}_\mu(\widetilde{\Phi}^* h^{-\frac{1}{4}})\right)\right. \\ &\qquad\qquad \left. - h^{\mu\nu}\mathcal{D}_\mu(\widetilde{\Phi}h^{-\frac{1}{4}})\bar{\mathcal{D}}_\nu(\widetilde{\Phi}^* h^{-\frac{1}{4}})\right] \end{aligned} \tag{8.34}$$

The fundamental fields of Eq. (8.34) are now $\{\widetilde{\Phi}, \widetilde{\Phi}^*, A_\mu, h^{\mu\nu}, \tau^\mu\}$. The total variation of the action Eq. (8.34) in this case can be expressed as

$$\delta\widetilde{S} = \int dt d^2x \left[-\widetilde{P}_{\mu\nu}\delta h^{\mu\nu} + \widetilde{R}_\mu \delta\tau^\mu - \widetilde{J}^\mu \delta A_\mu + \delta\widetilde{\Phi}^* \mathcal{R}\widetilde{\Phi} + \delta\widetilde{\Phi}\left(\mathcal{R}\widetilde{\Phi}\right)^*\right] \tag{8.35}$$

with

$$
\begin{aligned}
\widetilde{P}_{\mu\nu} &= \frac{\sqrt{h}}{2} h_{\mu\nu} \widetilde{\mathcal{L}} + \sqrt{h} \mathcal{D}_\mu (\widetilde{\Phi} h^{-\frac{1}{4}}) \bar{\mathcal{D}}_\nu (\widetilde{\Phi}^* h^{-\frac{1}{4}}) - \frac{1}{4} h_{\mu\nu} \left(\widetilde{\Phi}^* \mathcal{R} \widetilde{\Phi} + \widetilde{\Phi} (\mathcal{R} \widetilde{\Phi})^* \right) \\
\widetilde{R}_\mu &= imh^{\frac{1}{4}} \left(\widetilde{\Phi}^* \mathcal{D}_\mu (\widetilde{\Phi} h^{-\frac{1}{4}}) - \widetilde{\Phi} \bar{\mathcal{D}}_\mu (\widetilde{\Phi}^* h^{-\frac{1}{4}}) \right) \\
\widetilde{J}^\mu &= 2m^2 \widetilde{\Phi} \widetilde{\Phi}^* v^\mu + imh^{\frac{1}{4}} \left(\widetilde{\Phi}^* \partial^\mu (\widetilde{\Phi} h^{-\frac{1}{4}}) - \widetilde{\Phi} \partial^\mu (\widetilde{\Phi}^* h^{-\frac{1}{4}}) \right) \\
\mathcal{R} \widetilde{\Phi} &= \left(h^{\frac{1}{4}} \mathcal{D} h^{-\frac{1}{4}} \right) \widetilde{\Phi} = \left[h^{\frac{1}{4}} \left(2im\tau^\mu \mathcal{D}_\mu + im \nabla_\mu \tau^\mu + h^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu \right) h^{-\frac{1}{4}} \right] \widetilde{\Phi} \\
\left(\mathcal{R} \widetilde{\Phi} \right)^* &= \left(h^{\frac{1}{4}} \bar{\mathcal{D}} h^{-\frac{1}{4}} \right) \widetilde{\Phi}^* = \left[h^{\frac{1}{4}} \left(-2im\tau^\mu \mathcal{D}_\mu - im \nabla_\mu \tau^\mu + h^{\mu\nu} \bar{\mathcal{D}}_\mu \bar{\mathcal{D}}_\nu \right) h^{-\frac{1}{4}} \right] \widetilde{\Phi}^*
\end{aligned}
\tag{8.36}
$$

We now find that Eq. (8.35) vanishes under

$$
\delta_\Lambda \widetilde{\Phi} = \Lambda \widetilde{\Phi} \,, \; \delta_\Lambda \widetilde{\Phi}^* = \Lambda \widetilde{\Phi}^* \,, \tag{8.37}
$$

$$
\delta_\Lambda h^{\mu\nu} = -2\Lambda h^{\mu\nu} \,, \; \delta_\Lambda \tau^\mu = -2\Lambda \tau^\mu \,. \tag{8.38}
$$

Thus the action Eq. (8.34) is invariant under Weyl transformations. Considering the on-shell invariance of Eq. (8.34) under Weyl transformations ($\delta\widetilde{\Phi} = 0 = \delta\widetilde{\Phi}^*$), we find

$$
\begin{aligned}
0 = \delta_\Lambda \widetilde{S} &= \int dt d^2 x \sqrt{h} [-\widetilde{P}_{\mu\nu} \delta_\Lambda h^{\mu\nu} + \widetilde{R}_\mu \delta_\Lambda \tau^\mu] \\
&= \int dt d^2 x \sqrt{h} 2\Lambda [\widetilde{T}^\mu{}_\mu] \,,
\end{aligned}
\tag{8.39}
$$

where

$$
\widetilde{T}^\mu{}_\mu = -\frac{1}{2} (\widetilde{\Phi}^* \left(h^{-\frac{1}{4}} \mathcal{D} h^{-\frac{1}{4}} \right) \widetilde{\Phi} + \widetilde{\Phi} \left(h^{-\frac{1}{4}} \bar{\mathcal{D}} h^{-\frac{1}{4}} \right) \widetilde{\Phi}^*) \,. \tag{8.40}
$$

We have denoted $(2\widetilde{T}^0{}_0 + \widetilde{T}^i{}_i)$ as $\widetilde{T}^\mu{}_\mu$ in the above equations. It is evident from Eq. (8.39) that the on-shell Weyl invariance of Eq. (8.34) can be restored provided

$$
\widetilde{T}^\mu{}_\mu = 0 \,. \tag{8.41}
$$

We have thus demonstrated that the 2 + 1 dimensional Schrödinger field on the Newton-Cartan background can be used to investigate its invariance under both diffeomorphisms and Weyl transformations (the latter by densitizing the Schrödinger fields). This will be particularly useful in investigating both trace and diffeomorphism anomalies in the following section.

8.4 Derivation of the gravitational anomalies

Relativistic gravitational anomalies using Fujikawa's approach can be calculated in a covariant notation in a local plane wave basis. In the NR case, we do need to distinguish between time and space in both the regulator as well as the plane waves. We thus need to make use of a specific set of coordinates in our calculation. The adapted coordinates [19] provides a faithful representation of the NC structure (in the absence of torsion). Let Greek indices $\mu, \nu, \cdots$ denote spacetime coordinates, Latin indices $i, j, \cdots$ denote spatial coordinates and 0 represent the coordinate for time. Then the NC system of equations for the metric can be realized through the following choice

$$\tau_0 = 1 = \tau^0 \,, \quad \tau_i = 0 \,, \quad h^{0\mu} = 0 \tag{8.42}$$

Eq. (8.42) represents our choice of time. The normalization of τ_μ Eq. (5.3) allows us to make the choice given in Eq. (8.42). Since A_μ is a gauge field, it is naturally left unspecified. The adapted coordinate system may not be appropriate in the presence of torsion as τ_i in general cannot vanish. In adapted coordinates v^μ and ψ in Eq. (8.28) can be decomposed into temporal and spatial parts,

$$\begin{aligned}
&v^0 = \tau^0, \; v^i = \tau^i - h^{ij} A_j \\
&\psi = \bar{\phi} + \phi \\
&\phi = \tau^0 A_0, \; \bar{\phi} = \tau^i A_i - \frac{1}{2} h^{ij} A_i A_j
\end{aligned} \tag{8.43}$$

where v^0, v^i, ϕ and $\bar{\phi}$ are Milne invariant quantities.Using Eq. (5.6) and Eq. (8.42), we have the following non-vanishing components for the connection

$$\begin{aligned}
\Gamma^i_{jk} &= \left\{ {}^{\,i}_{jk} \right\} , \\
\Gamma^i_{0j} &= \frac{h^{ik}}{2} \left(\partial_j h_{k0} + \partial_0 h_{kj} - \partial_k h_{0j} + \partial_k A_j - \partial_j A_k \right) \,, \quad \Gamma^i_{0i} = \frac{h^{ik}}{2} \partial_0 h_{ik} \,, \\
\Gamma^i_{00} &= \frac{h^{ik}}{2} \left(2\partial_0 h_{k0} - \partial_k h_{00} \right) + h^{ik} \left(\partial_k \phi - \partial_0 A_k \right) \,,
\end{aligned} \tag{8.44}$$

where $\left\{ {}^{\,i}_{jk} \right\}$ represents the "Christoffel" component of the connection for the spatial metric (the second term of Eq. (5.6)). Notably $h_{0\mu}$ need not vanish in adapted coordinates, and therefore τ^i can exist. Using Eq. (5.4) and Eq. (8.42) we find that $h_{\mu\nu}$ and τ^μ satisfy the following relations

$$\begin{aligned}
h_{ij} \tau^j &= -h_{i0}, \quad \tau^i = -h^{ij} h_{j0} \,, \\
h_{00} &= -h_{0j} \tau^j = \tau^i h_{ij} \tau^j \,.
\end{aligned} \tag{8.45}$$

It can now be seen that the mass dimension of the connection components in Eq. (8.44) are not the same. The first line of Eq. (8.44) has mass dimension 1, the second line has mass dimension 2, while the last line has mass dimension 3. This reflects the $z = 2$ invariance of the background. However, Ricci and Riemann tensor components have a uniform mass dimension as a consequence. For instance

$$R_{00} = \Gamma^i_{00,i} - \Gamma^i_{0i,0} + \Gamma^i_{ij} \Gamma^j_{00} - \Gamma^i_{0j} \Gamma^j_{0i} \,, \tag{8.46}$$

has mass dimension 4, while R_{ij} has mass dimension 2. In the following subsection we will derive the trace and diffeomorphism anomalies using this coordinate system.

8.4.1 The trace anomaly

To derive the trace anomaly, we consider the action in (Eq. (8.34)), which can be expressed as

$$\begin{aligned} S &= \int dtd^2x\widetilde{\Phi}^*\tilde{\mathcal{Q}}\widetilde{\Phi} \\ &= \int dtd^2x\widetilde{\Phi}^*\left(h^{\frac{1}{4}}\mathcal{D}h^{-\frac{1}{4}}\right)\widetilde{\Phi}\,, \end{aligned} \tag{8.47}$$

where $\widetilde{\Phi}$ and $\widetilde{\Phi}^*$ are the fundamental fields and $\mathcal{D}$ is the Hermitian operator present in Eq. (8.30). The path integral is given by

$$Z = \int \mathcal{D}\widetilde{\Phi}\mathcal{D}\widetilde{\Phi}^* e^{iS\left[\widetilde{\Phi},\widetilde{\Phi}^*,\tau^\mu,h^{\mu\nu},A_\mu\right]}\,. \tag{8.48}$$

Using Eq. (8.37), we find that the invariance of Eq. (8.48) under Weyl transformations of the fields $\widetilde{\Phi}$ and $\widetilde{\Phi}^*$ results in the following anomalous Ward identity

$$\left\langle \Lambda\sqrt{h}\widetilde{T}^\mu{}_\mu \right\rangle_{\widetilde{\Phi}\widetilde{\Phi}^*} = \langle \mathrm{Tr}J\rangle_{\widetilde{\Phi}\widetilde{\Phi}^*}\,, \tag{8.49}$$

where $\langle\cdots\rangle_{\widetilde{\Phi}\widetilde{\Phi}^*}$ denotes the path integral average with respect to the variables $\widetilde{\Phi}$ and $\widetilde{\Phi}^*$. To proceed, we regulate the trace occurring in Eq. (8.49)

$$\langle \mathrm{Tr}J\rangle_{\widetilde{\Phi}\widetilde{\Phi}^*} \to \lim_{M\to\infty} \mathrm{Tr}Je^{\frac{\mathcal{R}}{M^2}}\,. \tag{8.50}$$

The Jacobian and the regulator to be used can be determined by comparing Eq. (8.47) with Eq. (8.20). The regulator to be used can be identified from Eq. (8.34),

$$\mathcal{R} = h^{\frac{1}{4}}\mathcal{D}h^{-\frac{1}{4}}\,, \tag{8.51}$$

and we have $J = \Lambda(x)$ (since $\mathbf{T}$ is a constant). The regulated trace which needs to be evaluated is now given by

$$\lim_{M\to\infty} \mathrm{Tr}\Lambda(x)e^{\frac{\mathcal{R}}{M^2}} = \lim_{M\to\infty} \int \frac{d\omega}{2\pi}\int \frac{d^2k}{(2\pi)^2} e^{-i\omega t}e^{ikx}\left[\Lambda(x)e^{\frac{\mathcal{R}}{M^2}}\right]e^{i\omega t}e^{-ikx}\,. \tag{8.52}$$

Due to the use of flat space non-relativistic plane waves, we expand $\mathcal{R}$ in the basis of the adapted coordinates,

$$\mathcal{R} = h^{\frac{1}{4}}\left[2imv^0\partial_t + 2imv^i\partial_i + h^{ij}\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right) - im\mathcal{C}\right]h^{-\frac{1}{4}}\,, \tag{8.53}$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\mathcal{C}$ is given by

$$\mathcal{C} = -\nabla_i v^i + 2im\left(\bar{\phi} + \phi\right)\,. \tag{8.54}$$

We can now move the plane wave from the right of the regulator in Eq. (8.52) to the left. By further rescaling $k \to Mk$ and $\omega \to M^2\omega$ we have

$$\lim_{M\to\infty} \mathrm{Tr}\Lambda(x)e^{\frac{\mathcal{R}}{M^2}} = \lim_{M\to\infty} M^4 \int \frac{d\omega}{2\pi}\int \frac{d^2k}{(2\pi)^2}\Lambda(x)e^{\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}}\,, \tag{8.55}$$

where the operator in the exponent now takes the form

$$\frac{\mathcal{R}(Mk, M^2\omega)}{M^2} = -k^2 - 2mv^0\omega + \frac{1}{M}\left(ik_i\Gamma^i - 2ik_i\partial^i + 2mk_iv^i - 2ih^{\frac{1}{4}}k_i\partial^i(h^{-\frac{1}{4}})\right) + \frac{1}{M^2}\left(\Delta - im\mathcal{C} + h^{\frac{1}{4}}\Delta h^{-\frac{1}{4}} + 2h^{\frac{1}{4}}\partial^l(h^{-\frac{1}{4}})\partial_l\right). \tag{8.56}$$

In Eq. (8.56) we have used the following definitions,

$$\Gamma^i = h^{mn}\Gamma^i_{mn}, \quad k^2 = k_ik_jh^{ij}, \\ \Delta = \partial^i\partial_j - \Gamma^i\partial_i + 2imv^0\partial_t + 2imv^i\partial_i. \tag{8.57}$$

At this stage we can factor out $e^{-2mv^0\omega}$ from $e^{\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}}$ since it is a constant ($v^0 = 1$ in adapted coordinates). Following this, the ω integral can be easily evaluated

$$\int_0^\infty \frac{d\omega}{2\pi}e^{-2m\omega} = \frac{1}{4\pi m}. \tag{8.58}$$

Concerning the k integral, we need to use the BCH expansion to factor out e^{-k^2} from $e^{\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}}$. By labelling $A = -k^2$ and B as the M dependent terms of $\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}$, we can write

$$e^{A+B} = e^Ae^E, \tag{8.59}$$

where E is given by

$$E = B - \frac{[A,B]}{2} + \frac{[A,[A,B]]}{6} + \frac{[B,[A,B]]}{12} - \frac{[A,[B,[A,B]]]}{24} - \frac{[A,[A,[A,B]]]}{24} \\ + \frac{[A\,[A,[A,[A,B]]]]}{120} + \frac{[A\,[A,[B,[A,B]]]]}{120} - \frac{[A\,[B,[B,[A,B]]]]}{240} \\ + \frac{[B\,[A,[B,[A,B]]]]}{180} - \frac{[B\,[B,[B,[A,B]]]]}{720} + \frac{[B\,[A,[A,[A,B]]]]}{240} + \cdots. \tag{8.60}$$

The ellipsis in Eq. (8.60) refers to the fifth order onward terms in BCH expansion. The commutators in Eq. (8.60) contain all contributions up to M^{-4} resulting from the BCH expansion, and their expressions have been provided in Eq. (8.101). From Eq. (8.101) we see that all terms with even powers of M^{-1} contain an even number of k's, and likewise all terms with odd powers of M^{-1} contain an odd number of k's. This property will hold to all orders in the BCH expansion.

Since E contains M^{-1} terms, we expand Eq. (8.59) up to fourth order

$$e^{A+B} = e^A\left(1 + E + \frac{E^2}{2} + \frac{E^3}{3!} + \frac{E^4}{4!}\right) + \mathcal{O}(E^5). \tag{8.61}$$

Eq. (8.61) now contains all terms up to M^{-4} which can contribute to the anomaly. We can now ignore all terms with free derivatives, as they cannot contribute to the anomaly. It will also be useful to separate those terms which do contain derivatives acting on $h^{-\frac{1}{4}}$, from those that do not. We thus write Eq. (8.61) as

$$\begin{aligned} e^{A+B} &= e^{A}\left(1+E+\frac{E^2}{2}+\frac{E^3}{3!}+\frac{E^4}{4!}\right)+\mathcal{O}(E^5) \\ &\approx e^{A}\left(1+\frac{\mathcal{B}_1}{M}+\frac{\mathcal{B}_2}{M^2}+\frac{\mathcal{B}_3}{M^3}+\frac{\mathcal{B}_4}{M^4}+\mathcal{H}(h^{-\frac{1}{4}})+\mathcal{O}(M^{-5})\right). \end{aligned} \tag{8.62}$$

The $\approx$ symbol in Eq. (8.62) indicates that we have dropped all terms with free derivatives. $\mathcal{H}(h^{-\frac{1}{4}})$ contains all terms with $\partial(h^{-\frac{1}{4}})$, with powers up to M^{-4}. The $\mathcal{B}_i$ represent the order M^{-i} contributions to the anomaly (which do not contain $\partial(h^{-\frac{1}{4}})$) . With Eq. (8.62), we have the following expression

$$e^{\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}} = e^{-2m\omega}e^{-k^2}\left(1+\frac{\mathcal{B}_1}{M}+\frac{\mathcal{B}_2}{M^2}+\frac{\mathcal{B}_3}{M^3}+\frac{\mathcal{B}_4}{M^4}+\mathcal{H}(h^{-\frac{1}{4}})\right), \tag{8.63}$$

which will be needed to evaluate the integrals. Upon substituting Eq. (8.63) and Eq. (8.58) in Eq. (8.55), we get

$$\begin{aligned} &\lim_{M\to\infty} \mathrm{Tr}\Lambda(x)e^{\frac{\mathcal{R}}{M^2}} \\ &= \lim_{M\to\infty} M^4\frac{1}{4\pi m}\int\frac{d^2k}{(2\pi)^2}\Lambda(x)e^{-k^2}\left(1+\frac{\mathcal{B}_1}{M}+\frac{\mathcal{B}_2}{M^2}+\frac{\mathcal{B}_3}{M^3}+\frac{\mathcal{B}_4}{M^4}+\mathcal{H}(h^{-\frac{1}{4}})\right) \end{aligned} \tag{8.64}$$

Eq. (8.64) can now be evaluated via the following Gaussian integrals

$$\begin{aligned} &\int d^2k\, e^{-k^2} = \sqrt{h}\pi\,, \int d^2k\, e^{-k^2}k_ik_j = \frac{1}{2}\sqrt{h}\pi h_{ij} \\ &\int d^2k\, e^{-k^2}k_ik_jk_mk_n = \frac{1}{4}\sqrt{h}\pi\,(h_{ij}h_{mn}+h_{im}h_{nj}+h_{in}h_{mj}) \\ &\int d^2k\, e^{-k^2}\,k_ik_j\cdots k_{2n-1}k_{2n} = \frac{1}{2^n}\sqrt{h}\pi\,((2n-1)!!\text{ permutations of }h_{ij}\cdots h_{2n-1\,2n})\,. \end{aligned} \tag{8.65}$$

The 'k integrals' vanish under symmetric integration whenever there are an odd number of k's in the integrand. Thus $\mathcal{B}_1$ and $\mathcal{B}_3$ vanish following symmetric integration. $\mathcal{H}(h^{-\frac{1}{4}})$ also vanishes following symmetric integration, which could have been anticipated following the cyclicity of trace [4]. The integral

$$\int d^2ke^{-k^2}\left(1+\frac{\mathcal{B}_2}{M^2}\right), \tag{8.66}$$

is non-vanishing. These terms however can be eliminated by regularization, and do not contribute in the final expression for the anomaly. For example, within the Pauli-Villars scheme one can include additional copies of the PV fields with

[4]The cyclicity of trace works here because the Jacobian does not involve any free derivatives.

coefficients chosen so as to cancel out these M dependent terms. Thus these terms can be ignored as well. Since the integral of $\mathcal{B}_2$ is somewhat instructive, we have provided the terms contained in its integrand in Eq. (8.102), using which we have the following result

$$\int d^2ke^{-k^2}\frac{\mathcal{B}_2}{M^2} = \sqrt{h}\pi\left(\frac{1}{6}R_{ij}h^{ij} + 2m^2\phi\right) . \tag{8.67}$$

The only contribution to the anomaly comes from the term $\mathcal{B}_4$, and Eq. (8.64) reduces to

$$\lim_{M\to\infty} \mathrm{Tr}\Lambda(x)e^{\frac{\mathcal{R}}{M^2}} = \frac{1}{4\pi m}\int\frac{d^2k}{(2\pi)^2}\Lambda(x)e^{-k^2}\mathcal{B}_4 \tag{8.68}$$

The individual terms contained in $\mathcal{B}_4$ have been provided in Eq. (8.103), and the resulting k integral works out to give

$$\int d^2ke^{-k^2}\mathcal{B}_4 = \sqrt{h}\pi\left(\frac{1}{180}(R_{ijmn}R^{ijmn} - R_{ij}R^{ij} + \Box R_{ij}h^{ij})\right.$$
$$\left. +2m^4\phi^2 + \frac{m^2}{3}(\phi R_{ij}h^{ij} + R_{00}\tau^0\tau^0)\right) . \tag{8.69}$$

Substituting Eq. (8.69) in Eq. (8.68), we get the following expression for the candidate anomaly,

$$\lim_{M\to\infty} \mathrm{Tr}\Lambda(x)e^{\frac{\mathcal{R}}{M^2}} = \frac{\sqrt{h}\Lambda(x)}{m(4\pi)^2}\left(\frac{1}{180}(R_{ijmn}R^{ijmn} - R_{ij}R^{ij} + \Box R_{ij}h^{ij})\right.$$
$$\left. +2m^4\phi^2 + \frac{m^2}{3}(\phi R_{ij}h^{ij} + R_{00}v^0v^0)\right) \tag{8.70}$$

While the calculation leading to this result is considerably involved, we note the following points related to the derivation and the above result. The term $R_{00}v^0v^0$ results due to both the single derivative operator ∂_t and $im\mathcal{C}$ contained in Eq. (8.53), following the BCH expansion. If A_μ were absent in our derivation, then so too would all the terms in the second line of Eq. (8.70), providing only the curvature squared results already noted in the literature. The choice of $\tau_\mu = (1, 0)$ and the absence of $h^{0\mu}$ in adapted coordinates affects the expressions of $\mathcal{C}$, the Ricci and Riemann tensors, as well as the final result. The absence of terms involving τ^i and by extension v^i in the final answer is thus a coordinate artifact which reflects our choice of time for the hypersurface. Remarkably, all imaginary terms cancel out in the calculation leading to Eq. (8.70). Both these points may also be noted to be the case with Eq. (8.67).

We can now determine which right hand side terms of Eq. (8.70) represent the true anomaly. A local counterterm involving $(R_{ij}h^{ij})^2$ can be included in the effective action to eliminate the term $\Box R_{ij}h^{ij}$, and hence is also not part of the result. Further, since $(R_{ijmn}R^{ijmn} - R_{ij}R^{ij})$ is constructed out of the $2d$ spatial metric, we can use $R_{ijmn} = \frac{1}{2}(R_{ij}h^{ij})(h_{im}h_{jn} - h_{in}h_{jm})$ to write

$$R_{ijmn}R^{ijmn} - R_{ij}R^{ij} = \frac{1}{2}\left(R_{ij}h^{ij}\right)^2 . \tag{8.71}$$

Note that Eq. (8.71) is valid for NC backgrounds which satisfy the Frobenius condition. The terms $2m^4\phi^2$ and $\phi R_{ij}h^{ij}$ notably violate U(1) invariance. This can in principle be allowed since there can exist 'U(1) gravitational anomalies'. This follows from the U(1) invariance of the action as well as the field Φ having a U(1) transformation. For relativistic theories with both gauge and gravitational fields in 4 dimensions, U(1) violating terms in the diffeomorphism anomaly arise and the gauge current is also anomalous (and is related to the Pontryagin density). However, one can find a counterterm to make the gauge current anomaly free, which in turn leads to the diffeomorphism anomaly being U(1) invariant. Thus a situation similar to that of mixed gravitational anomalies in relativistic theories may arise here. Here however, the anomalous current $\langle J^\mu \rangle$ is also a gravitational anomaly due to its presence in the connection. Thus Eq. (8.70) provides the following expression for the trace anomaly

$$\mathrm{An}_\Lambda = \frac{\sqrt{h}\Lambda(x)}{m(4\pi)^2}\left(\frac{1}{360}(R_{ij}h^{ij})^2 + 2m^4\phi^2 + \frac{m^2}{3}(\phi R_{ij}h^{ij} + R_{00}v^0v^0)\right). \tag{8.72}$$

Using Eq. (8.72) and Eq. (8.50), we can now write the covariant result as

$$\left\langle 2\widetilde{T}^0{}_0 + \widetilde{T}^i{}_i \right\rangle = \frac{1}{m(4\pi)^2}\left(\frac{1}{360}(R_{\mu\nu}h^{\mu\nu})^2 + 2m^4\psi^2 + \frac{m^2}{3}(\psi R_{\mu\nu}h^{\mu\nu} + R_{\mu\nu}v^\mu v^\nu)\right) \tag{8.73}$$

We note that this covariant result has been inferred from the result by correcting for our choice of coordinates discussed earlier (the choice of $\tau_\mu = (1,0,0,0)$ and $h^{0\mu} = 0$) and by requiring that the result should be Milne invariant. The result in Eq. (8.72) as well as the regulator in Eq. (8.53) were Milne invariant (within adapted coordinates). Further, since 'Milne gravitational anomalies' do not exist[5], we seek a Milne invariant expression Eq. (8.73).

We note that our calculation demonstrates that the trace anomaly only arises in odd dimensions. Since $z = 2$ and all BCH expansion terms involve an even(odd) number of k's for terms with an even(odd) power of M^{-1}, the anomalies can only occur when there are an even number of spatial dimensions. Thus NC trace anomalies always arise in odd spacetime dimensions. Our result concerns NC backgrounds without torsion. In the general case we would have instead,

$$R_{ijmn}R^{ijmn} - R_{ij}R^{ij} = \frac{1}{2}(-\bar{E}_4 + 3\bar{C}^2) \tag{8.74}$$

where E_4 and C^2 represent the four dimensional Euler density and the square of the Weyl tensor respectively as follows,

$$\begin{aligned} E_4 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,, \\ C^2 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\,, \end{aligned} \tag{8.75}$$

[5] As the field Φ does not transform under Milne transformations, there can be no corresponding anomaly even though the action itself is Milne invariant

while the overbar implies that these tensors are contracted only with the (two dimensional) spatial metric $h^{\alpha\beta}$. The general result, following Eq. (8.73), will then be modified to

$$\left\langle 2\widetilde{T}^0{}_0 + \widetilde{T}^i{}_i \right\rangle = \frac{1}{m(4\pi)^2}\left(\frac{1}{360}\left(-\bar{E}_4 + 3\bar{C}^2\right) + 2m^4\psi^2 + \frac{m^2}{3}(\psi R_{\mu\nu}h^{\mu\nu} + R_{\mu\nu}v^\mu v^\nu)\right)$$
$$+ \text{additional terms}\,. \tag{8.76}$$

This result, apart from the τ^μ and A^μ dependent terms is in agreement with the results provided in [93,95]. The coefficients of the curvature squared terms are in addition identical to those derived using the heat kernel approach of [98].

8.4.2 The diffeomorphism anomaly

The diffeomorphism anomaly can be computed from Eq. (8.27) using the procedure of the previous subsection. The fundamental fields are now Φ and Φ^* with the following action

$$\begin{aligned} S &= \int dt d^2x \Phi^* \sqrt{h}\mathcal{D}\Phi \\ &= \int dt d^2x \Phi^* \mathbf{T}\mathcal{Q}\Phi\,. \end{aligned} \tag{8.77}$$

The path integral in this case is given by,

$$Z = \int \mathcal{D}\Phi\mathcal{D}\Phi^* e^{iS[\Phi,\Phi^*,\tau^\mu,h^{\mu\nu},A_\mu]} \tag{8.78}$$

Using Eq. (8.31), the invariance of Eq. (8.78) under $\delta\Phi = \pounds_\xi\Phi$ and $\delta\Phi^* = \pounds_\xi\Phi^*$ results in the following anomalous Ward identity

$$\left\langle -\sqrt{h}\xi^\mu\left(\nabla_\nu T^\nu{}_\mu + J^\nu\nabla_{[\mu}A_{\nu]} - \frac{1}{2}R_\nu\nabla_\mu\tau^\nu\right)\right\rangle_{\Phi\Phi^*} = \langle \mathrm{Tr}J\rangle_{\Phi\Phi^*}\,. \tag{8.79}$$

From Eq. (8.20) we choose $\mathcal{R} = \mathcal{D}$, which ensures that it is symmetric[6]. In this case $\mathbf{T} = \sqrt{h}$, and hence the Jacobian to be considered is,

$$\begin{aligned} J &= \xi^\mu\partial_\mu + \frac{1}{2\sqrt{h}}\pounds_\xi\sqrt{h} \\ &= \xi^\mu\partial_\mu + \frac{1}{2\sqrt{h}}\xi^\mu\partial_\mu\sqrt{h} + \partial_\mu\xi^\mu\,. \end{aligned} \tag{8.80}$$

Thus the regulated trace takes the following form

$$\begin{aligned} &\lim_{M\to\infty}\mathrm{Tr}Je^{\frac{\mathcal{R}}{M^2}} \\ &= \lim_{M\to\infty}\int\frac{d\omega}{2\pi}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega t}e^{ikx}\left[\left(\xi^\mu\partial_\mu + \frac{\xi^\mu\partial_\mu\sqrt{h}}{2\sqrt{h}} + \partial_\mu\xi^\mu\right)e^{\frac{\mathcal{R}}{M^2}}\right]e^{i\omega t}e^{-ikx}\,. \end{aligned} \tag{8.81}$$

[6] By symmetric we mean that $\int\Phi^*\mathcal{D}\Phi = \int\Phi\bar{\mathcal{D}}\Phi^*$

Evaluating this expression would formally result in considering an expansion up to M^{-6}. In taking the plane wave ($e^{i\omega t}e^{-ikx}$) to the left, it gets acted upon by both the Jacobian and the regulator. The action of the Jacobian on $e^{i\omega t}$ now produces the term $i\xi^0\omega$. By rescaling $\omega \to M^2\omega$, we end up with a factor of M^6 outside the above integral, requiring a BCH expansion up to M^{-6} for determining the anomaly.

However, having chosen a symmetric regulator we can avoid this cumbersome calculation. First, we note the following identity which holds for any symmetric regulator $\widetilde{\mathcal{R}}$ [90]

$$\mathrm{Tr}\left(\xi^\mu\partial_\mu + \frac{1}{2}\partial_\mu\xi^\mu\right)e^{\widetilde{\mathcal{R}}} = 0\,. \tag{8.82}$$

Using the expressions for $\Gamma^i_{\mu i} = \frac{1}{\sqrt{h}}\partial_\mu\sqrt{h}$ and $\Gamma^0_{\mu\nu} = 0$ (in adapted coordinates) and Eq. (8.82), we can simplify Eq. (8.81) to

$$\lim_{M\to\infty}\mathrm{Tr}Je^{\frac{\mathcal{R}}{M^2}} = \lim_{M\to\infty}\int\frac{d\omega}{2\pi}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega t}e^{ikx}\left[\frac{1}{2}\left(\nabla_\mu\xi^\mu\right)e^{\frac{\mathcal{R}}{M^2}}\right]e^{i\omega t}e^{-ikx}\,, \tag{8.83}$$

and hence we don't have to deal with any free derivatives due to the Jacobian. Moving the plane wave past the regulator and rescaling $k \to Mk$ and $\omega \to M^2\omega$ results in

$$\lim_{M\to\infty}\mathrm{Tr}Je^{\frac{\mathcal{R}}{M^2}} = \lim_{M\to\infty}M^4\int\frac{d\omega}{2\pi}\int\frac{d^2k}{(2\pi)^2}\frac{1}{2}\left(\nabla_\mu\xi^\mu\right)e^{\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}} \tag{8.84}$$

We now need to factor out e^{-k^2} and $e^{-2m\omega}$ from $e^{\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}}$ using the BCH expansion, as in the previous section, up to M^{-4} terms. Since the regulator of this subsection differs from that of the previous one only by $\partial(h^{-\frac{1}{4}})$ terms, the following factored expression is easily determined from Eq. (8.63)

$$e^{\frac{\mathcal{R}(Mk,M^2\omega)}{M^2}} = e^{-2m\omega}e^{-k^2}\left(1 + \frac{\mathcal{B}_1}{M} + \frac{\mathcal{B}_2}{M^2} + \frac{\mathcal{B}_3}{M^3} + \frac{\mathcal{B}_4}{M^4}\right)\,. \tag{8.85}$$

Only the $\mathcal{B}_4$ term contributes to the anomaly, and we have the following expression for the candidate anomaly

$$\begin{aligned}\lim_{M\to\infty}\mathrm{Tr}Je^{\frac{\mathcal{R}}{M^2}} &= \frac{1}{2}\left(\nabla_\mu\xi^\mu\right)\int\frac{d\omega}{2\pi}e^{-2m\omega}\int\frac{d^2k}{(2\pi)^2}e^{-k^2}\mathcal{B}_4\\ &= \frac{\sqrt{h}\left(\nabla_\mu\xi^\mu\right)}{(4\pi)^2m}\left(\frac{1}{360}\left((R_{ij}h^{ij})^2 + \Box(R_{ij}h^{ij})\right) + 2m^4\phi^2 + \frac{m^2}{3}(\phi R_{ij}h^{ij} + R_{00}v^0v^0)\right)\end{aligned} \tag{8.86}$$

where we have simplified the curvature squared expression by making use of Eq. (8.71) . The terms from Eq. (8.86) which contribute to the anomaly must satisfy the same criteria as in the case for the trace anomaly. Adopting the covariant notation as in the case of the trace anomaly, the result for the diffeomorphism anomaly in this case is

$$\mathrm{An}_\xi = -\frac{\sqrt{h}}{m(4\pi)^2}\xi^\mu\nabla_\mu\left(\frac{1}{720}(R_{\mu\nu}h^{\mu\nu})^2 + m^4\psi^2 + \frac{m^2}{6}(\psi R_{\mu\nu}h^{\mu\nu} + R_{\mu\nu}v^\mu v^\nu)\right). \tag{8.87}$$

Eq. (8.87) and Eq. (8.79) now provide the following expression for the diffeomorphism anomaly

$$\left\langle \nabla_\nu T^\nu{}_\mu + J^\nu\nabla_{[\mu}A_{\nu]} - \frac{1}{2}R_\nu\nabla_\mu\tau^\nu\right\rangle$$
$$= \nabla_\mu\left(\frac{1}{720(4\pi)^2 m}(R_{\alpha\beta}h^{\alpha\beta})^2 + \frac{m^3}{16\pi^2}\psi^2 + \frac{m}{96\pi^2}(\psi R_{\mu\nu}h^{\mu\nu} + R_{\mu\nu}v^\mu v^\nu)\right) \tag{8.88}$$

We emphasize that all currents occurring on the left hand side of Eq. (8.88) correspond to the gravitational fields of the NC background. We note that most of the previous results for the trace anomaly (based on DLCQ) indicate a one-to-one correspondence of the $2+1$ dimensional result of the NC background with $3+1$ dimensional result of relativistic backgrounds. Were this to actually be true for all gravitational anomalies, one would in fact naively expect there to be no diffeomorphism anomaly for the Schrödinger field in $2+1$ dimensions. In deriving this result, we have demonstrated that this is not the case. The presence of a diffeomorphism anomaly allows for several consequences in condensed matter systems with boundaries. In particular we note that this could be relevant in providing the entanglement entropy of Quantum Hall systems on curved backgrounds with boundaries [75], where the Schrödinger field is present in the low energy effective action.

8.5 A c-theorem condition

The coefficients of the trace anomaly are closely related to the renormalization group (RG) flow of a given theory. By applying the Wess-Zumino (WZ) consistency condition on the quantum effective action one can determine the constraints which relate the anomaly coefficients with the beta functions of the theory. Our treatment in this section will follow [77] where the consistency conditions for 2d and 4d relativistic CFTs are addressed. An investigation into the local RG flow of fields along these lines on the NC background was initiated in [99]. Here we will only consider the local RG flow of the $R_{\mu\nu}\tau^\mu\tau^\nu$ term of Eq. (8.88) due to marginal deformations. Our goal in this section will be to demonstrate that this term provides a c-theorem condition analogous to that of 2d CFTs. To begin with, let us consider the following renormalized partition function in the presence of sources,

$$Z[\mathcal{J}] = e^{i\mathcal{W}[\mathcal{J}]} = \int \mathcal{D}\widetilde{\Phi}\mathcal{D}\widetilde{\Phi}^* e^{iS[\widetilde{\Phi},\widetilde{\Phi}^*,\mathcal{J}]} \tag{8.89}$$

where $\mathcal{W}$ is the quantum effective action, which generates connected correlators associated with renormalized composite operators, and $\mathcal{J}$ denotes all the sources.

Here we will assume that $\mathcal{J}$ involves the independent background fields of the NC backround ($h^{\mu\nu}$, τ^{μ} and A_{μ}) and dimensionless coefficients g^I associated with certain marginal operator insertions $\mathcal{O}_I$ [7]. To investigate RG flows we first introduce the RG parameter μ. We can now define the RG time function $t = \ln\left(\frac{\mu}{\mu_0}\right)$, where μ_0 is some arbitrary reference scale, and the beta functions $\beta^I = \frac{\partial g^I}{\partial t}$ corresponding to the dimensionless parameters g^I. The flow is generated by $\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta^I\partial_I$, where we have further defined $\partial_I = \frac{\partial}{\partial g^I}$. In flat spacetime $\mathcal{W}$ satisfies the flow equation

$$\mathcal{D}\mathcal{W} = 0 \tag{8.90}$$

which is nothing but the Callan-Symanzik equation. The local RG concerns itself with the renormalizability of composite operators on curved backgrounds and hence the couplings are now functions of spacetime ($g^I = g^I(x,t)$). The local Callan-Symanzik equation under Weyl transformations is given by

$$\left(\Delta^W_\Lambda - \Delta^\beta_\Lambda\right)\mathcal{W} = \int\limits_{\mathcal{V}} dv B_\Lambda \tag{8.91}$$

where Λ is the local parameter involved in Weyl transformations, $\int\limits_{\mathcal{V}} dv$ is the integral involving the NC covariant volume element in $(2+1)$ dimensions and B_Λ is a local anomaly density involving derivatives of the NC fields and g^I. The variations Δ^W_Λ and Δ^β_Λ are defined as

$$\begin{aligned}
\Delta^W_\Lambda &= \int\limits_{\mathcal{V}} dv \left[2\Lambda h^{\mu\nu}\frac{\delta}{\delta h^{\mu\nu}} + 2\Lambda\tau^\mu\frac{\delta}{\delta\tau^\mu}\right] \\
\Delta^\beta_\Lambda &= \int\limits_{\mathcal{V}} dv\Lambda\beta^I\frac{\delta}{\delta g^I}
\end{aligned} \tag{8.92}$$

Eq. (8.91) reveals that at the critical point, where $\beta^I = 0$, B_Λ is simply the trace anomaly. Away from the critical point, we have additional dimension 4 terms involving the derivatives g^I. We can thus write Eq. (8.91) in the following way

$$\left(\Delta^W_\Lambda - \Delta^\beta_\Lambda\right)\mathcal{W} = \int\limits_{\mathcal{V}} dv\tau^\mu\tau^\nu\left[\Lambda\left(\frac{1}{2}\beta^\Phi R_{\mu\nu} - \frac{1}{2}\chi_{IJ}\partial_\mu g^I\partial_\nu g^J\right) - (\partial_\mu\Lambda)\,\omega_I\partial_\nu g^I + \cdots\right] \tag{8.93}$$

where β^Φ , χ_{IJ} and ω_I all depend on the coupling parameter g^I. The dots in Eq. (8.93) indicate the $(R_{\mu\nu}h^{\mu\nu})^2$ and other additional terms of dimension 4. These terms have been ignored since they will not be required in the following discussion. For simplicity we are also assuming that the NC background satisfies the Frobenius condition. Since Weyl transformations are Abelian, they satisfy the WZ consistency condition

$$\left[\Delta^W_\Lambda - \Delta^\beta_\Lambda\,,\,\Delta^W_{\Lambda'} - \Delta^\beta_{\Lambda'}\right]\mathcal{W} = 0 \tag{8.94}$$

[7] In general $\mathcal{J}$ also involves m^a associated with relevant operators $\mathcal{O}_a$, and vector sources $\mathcal{A}_\mu$ associated with certain currents $\mathcal{J}^\mu$ which the theory might possess

Using Eq. (8.93) and Eq. (8.94) gives the following expression

$$\left[\Delta^W_\Lambda - \Delta^\beta_\Lambda\,,\,\Delta^W_{\Lambda'} - \Delta^\beta_{\Lambda'}\right]\mathcal{W} = \int_{\mathcal{V}} dv\tau^\nu\left(\Lambda\partial_\nu\Lambda' - \Lambda'\partial_\nu\Lambda\right)\tau^\mu V_\mu = 0 \tag{8.95}$$

where

$$V_\mu = \partial_\mu\beta^\Phi - \left(\chi_{IJ}\beta^I - \beta^I\partial_I\omega_J - \omega_I\partial_J\beta^I\right)\partial_\mu g^J \tag{8.96}$$

Eq. (8.95) vanishes if V_μ vanishes. We see that Eq. (8.96) will vanish provided

$$\partial_J\beta^\Phi = \chi_{IJ}\beta^I - \beta^I\partial_I\omega_J - \omega_I\partial_J\beta^I \tag{8.97}$$

We now define the new function $\widetilde{\beta}^\Phi = \beta^\Phi + \omega_I\beta^I$, with which Eq. (8.97) becomes

$$\partial_J\widetilde{\beta}^\Phi = \chi_{IJ}\beta^I + \beta^I\left(\partial_J\omega_I - \partial_I\omega_J\right) \tag{8.98}$$

Contracting this equation with β^J now leads to the following result

$$\frac{\partial\widetilde{\beta}^\Phi}{\partial t} = \chi_{IJ}\beta^I\beta^J \tag{8.99}$$

This is a c-theorem condition satisfied by the coefficient of $R_{\mu\nu}\tau^\mu\tau^\nu$ on NC backgrounds with the Frobenius condition, which is analogous to the relation satisfied in 2d CFTs. At this point the proof of the c-theorem follows by establishing that the 'metric' χ_{IJ} is positive definite. In 2d CFTs, it can be shown that χ_{IJ} is essentially equivalent to 'Zamolodchikov's metric' $G_{IJ} = (x^2)^2\langle[\mathcal{O}_I(x)]\,[\mathcal{O}(0)]\rangle$, which further identifies $\widetilde{\beta}^\Phi$ with Zamolodchikov's c-function C [77]. Here the situation is not so straightforward since the marginal operators and the correlation functions they define differ from those of 2d CFTs. Our analysis would be incomplete without considering all the correlation functions of the theory, which would go outside the scope of the present work. We leave the investigation of this topic and the consistency conditions following the general form of B_Λ to future work.

8.6 BCH expansion terms

It will be convenient to introduce the following definitions

$$\begin{aligned}
\widetilde{\Gamma}^i &= \Gamma^i + 2im\left(h^{ij}A_j - \tau^i\right)\,,\\
G^m &= 2h^{-\frac{1}{4}}\partial^m h^{\frac{1}{4}}\,,\\
C^{ij} &= \Delta h^{ij}\,,\\
D^{lij} &= \partial^l h^{ij}\,,\; D_l{}^{ij} = \partial_l h^{ij}\,,\\
E^{ij} &= \Delta C^{ij} + 2imD^{lij}\partial_l\mathcal{C} + \left(\Delta + G^m\partial_m\right)G^l D_l{}^{ij} + G^l\partial_l C^{ij} - 2D^{lij}\partial_l\left(h^{-\frac{1}{4}}\Delta h^{\frac{1}{4}}\right)\,,\\
H^{lij} &= \partial^l C^{ij} + \Delta D^{lij} + D^{nij}\partial_n\widetilde{\Gamma}^l + \partial^l(G^m D_m{}^{ij}) + G^m A_m{}^{lij} - D_m{}^{ij}\partial^m G^l\,,\\
\Theta^{ijmn} &= D_k{}^{ij}D^{kmn}\,,\\
A^{ijmn} &= \partial^i D^{jmn}\,,\; A_{ij}{}^{mn} = \partial_i D_j{}^{mn}\,,\\
B^{ijmn} &= -2\Theta^{ijmn} + 2\left(A^{ijmn} + A^{jimn}\right)\,,
\end{aligned} \tag{8.100}$$

where $\partial^i \equiv h^{ij}\partial_j$ and $\mathcal{C}$ is as defined in Eq. (8.54). Then for

$$
\begin{aligned}
A &= -k_i k_j h^{ij} \\
B &= \frac{i}{M} k_i \left(\widetilde{\Gamma}^i - 2\partial^i - G^i\right) + \frac{1}{M^2}\left(\Delta - im\mathcal{C} + h^{\frac{1}{4}}\Delta h^{-\frac{1}{4}} + G^l\partial_l\right)
\end{aligned}
$$

the BCH terms which describe E in Eq. (8.60) can be expressed as

$$
\begin{aligned}
[A,B] &= -\frac{2i}{M}k_ik_jk_mD^{mij} + \frac{1}{M^2}k_ik_j\left(C^{ij} + 2D^{lij}\partial_l + G^lD_l{}^{ij}\right) \\
[A,[A,B]] &= \frac{2}{M^2}k_ik_jk_mk_n\Theta^{ijmn} \\
[B\,[A,B]] &= -\frac{4}{M^2}k_ik_jk_nk_mA^{mnij} - \frac{2i}{M^3}k_ik_jk_m\left(B^{ijml}\partial_l + H^{lij}\right) \\
&\quad + \frac{1}{M^4}k_ik_j\left(E^{ij} + B^{ijmn}\partial_m\partial_n + 2H^{lij}\partial_l\right) \\
[A,[A,[A,B]]] &= 0\,, \qquad [A,[A,[A,[A,B]]]] = 0\,, \qquad [B,[A,[A,[A,B]]]] = 0 \\
[A,[B,[A,B]]] &= -\frac{2i}{M^3}k_ik_jk_mk_nk_lB^{ijmp}\partial_ph^{nl} \\
&\quad + \frac{1}{M^4}k_ik_jk_mk_n\left(2H^{lij}D_l{}^{mn} + B^{ijlp}(A_{lp}{}^{mn} + 2D_l{}^{mn}\partial_p)\right) \\
[B,[B,[A,B]]] &= \frac{4}{M^4}k_ik_jk_mk_n\left[\left(B^{ijmp}D_p{}^{nl} - 2\partial^lA^{mnij} - \partial^nB^{ijml}\right)\partial_l\right. \\
&\quad \left. -\partial^nH^{mij} - \Delta A^{mnij} - \frac{1}{2}B^{ijml}\partial_l\widetilde{\Gamma}^n\right] + \frac{8i}{M^3}k_ik_jk_mk_nk_l\partial^nA^{mlij} \\
[A,[A,[B,[A,B]]]] &= \frac{2}{M^4}k_ik_jk_mk_nk_lk_kB^{ijpq}D_p{}^{mn}D_q{}^{lk} \\
[A,[B,[B,[A,B]]]] &= \frac{4}{M^4}k_ik_jk_mk_nk_pk_q\left(B^{ijmr}D_r{}^{nl} - 2\partial^lA^{mnij} - \partial^nB^{ijml}\right)D_l^{pq} \\
[B,[B,[B,[A,B]]]] &= \frac{16}{M^4}k_ik_jk_mk_nk_pk_q\partial^p\partial^qA^{mnij} \\
[B,[A,[B,[A,B]]]] &= -\frac{4}{M^4}k_ik_jk_mk_nk_pk_q\partial^p\left(B^{ijqr}D_r{}^{mn}\right) \qquad (8.101)
\end{aligned}
$$

The free derivatives contained in the BCH terms above, and thereby in E, are needed in computing $E^2\,, E^3$ ad E^4 in Eq. (8.61). With all expansions taken into consideration, we can drop the free derivative terms to arrive at Eq. (8.62). Only the terms $\mathcal{B}_2$ and $\mathcal{B}_4$ lead to non-trivial results following symmetric integration. By using $\partial_\alpha h^{ij} = -2\Gamma^{(i}_{\alpha k}h^{j)k}\,;\alpha = (0,i)$, the terms contained in $\mathcal{B}_2$ are, order by order, given by

$$
\begin{aligned}
&k^0 : -im\mathcal{C} \\
&k^2 : -\frac{1}{2}k_ik_j\left(C^{ij} + \widetilde{\Gamma}^i\widetilde{\Gamma}^j - 2\partial^i\widetilde{\Gamma}^j\right) \\
&k^4 : \frac{1}{3}k_ik_jk_mk_n\left(\Theta^{ijmn} + 2A^{ijmn}\right) - k_ik_jk_mk_n\widetilde{\Gamma}^iD^{jmn} \\
&k^6 : -\frac{1}{2}k_ik_jk_mk_nk_lk_kD^{lij}D^{kmn}\,. \qquad (8.102)
\end{aligned}
$$

Using these terms in Eq. (8.65) results in the expression of Eq. (8.67). The terms involved in $\mathcal{B}_4$ are considerably more involved and comprise the following

$$
\begin{aligned}
&k^0 : -\frac{1}{2}\left(m^2\mathcal{C}^2 + im\Delta\mathcal{C}\right)\\
&k^2 : k_ik_j \left[im\left(\frac{2}{3}\left(D^{lij}\partial_l\mathcal{C} + \partial^i\partial^j\mathcal{C}\right) + \frac{1}{2}\mathcal{C}\left(C^{ij} + \widetilde{\Gamma}^i\widetilde{\Gamma}^j\right) - \partial^i(\widetilde{\Gamma}^j\mathcal{C})\right)\right.\\
&\qquad\left. -\frac{1}{6}\Delta C^{ij} - \frac{1}{2}\widetilde{\Gamma}^i\Delta\widetilde{\Gamma}^j + \frac{1}{3}\left(\Delta\partial^i\widetilde{\Gamma}^j + \partial^i\Delta\widetilde{\Gamma}^j\right)\right]\\
&k^4 : k_ik_jk_mk_n\left[im\left(\mathcal{C}\left(\widetilde{\Gamma}^iD^{lmn} - \frac{1}{3}\left(\Theta^{ijmn} + 2A^{ijmn}\right)\right) - D^{jmn}\partial^i\mathcal{C}\right)\right.\\
&\quad + \frac{1}{6}\left(\partial^i\partial^jC^{mn} + \partial^i\Delta D^{jmn}\right) - \frac{1}{3}\widetilde{\Gamma}^i\left(\partial^jC^{mn} + \Delta D^{jmn}\right) - \frac{1}{2}\partial^i(D^{lmn}\partial_l\widetilde{\Gamma}^j)\\
&\quad + \frac{2}{3}\widetilde{\Gamma}^iD^{lmn}\partial_l\widetilde{\Gamma}^j + \frac{1}{6}\Delta\left(\Theta^{ijmn} + A^{ijmn}\right) + C^{ij}\left(\frac{1}{8}C^{mn} + \frac{1}{4}\widetilde{\Gamma}^m\widetilde{\Gamma}^n - \frac{1}{2}\partial^m\widetilde{\Gamma}^n\right)\\
&\quad - \frac{1}{12}D_l{}^{ij}\left(\Delta D^{jmn} + D^{lmn}\partial_l\widetilde{\Gamma}^j - 2\partial^lC^{mn}\right) + \frac{1}{24}\left(\widetilde{\Gamma}^i\widetilde{\Gamma}^j\widetilde{\Gamma}^m\widetilde{\Gamma}^n - B^{ijlp}A_{lp}{}^{mn}\right)\\
&\quad + \frac{1}{12}B^{ijml}\partial_l\widetilde{\Gamma}^n - \frac{1}{2}D^{mij}\Delta\widetilde{\Gamma}^n - \frac{1}{3}D^{lij}\partial_l\left(\partial^m\widetilde{\Gamma}^n\right)\\
&\quad\left. + \frac{1}{2}\left(\partial^i\widetilde{\Gamma}^j\right)\left(\partial^m\widetilde{\Gamma}^n\right) + \frac{2}{3}\widetilde{\Gamma}^m\partial^i\partial^j\widetilde{\Gamma}^n - \frac{1}{3}\partial^i\partial^j\partial^m\widetilde{\Gamma}^n - \frac{1}{2}\widetilde{\Gamma}^i\widetilde{\Gamma}^j\partial^m\widetilde{\Gamma}^n\right]\\
&k^6 : k_ik_jk_mk_nk_lk_k\left[\frac{1}{2}im\mathcal{C}D^{lij}D^{kmn} - \frac{1}{3}\left(\partial^lC^{ij} + \Delta D^{lij}\right)D^{kmn}\right.\\
&\quad - \frac{1}{6}C^{ij}\left(\Theta^{mnlk} + 2A^{lkmn} - 3\widetilde{\Gamma}^lD^{kmn}\right) - \frac{1}{9}\partial^l\partial^k\left(\Theta^{ijmn} + \frac{11}{5}A^{ijmn}\right)\\
&\quad + \left(\Theta^{ijmn} + 2A^{ijmn}\right)\left(\frac{1}{3}\partial^l\widetilde{\Gamma}^k - \frac{1}{6}\widetilde{\Gamma}^l\widetilde{\Gamma}^k\right) - \widetilde{\Gamma}^lD^{mij}\partial^n\widetilde{\Gamma}^k + \frac{1}{6}\widetilde{\Gamma}^l\widetilde{\Gamma}^k\widetilde{\Gamma}^mD^{nij}\\
&\quad + D^{rij}\partial_r\left(\frac{1}{2}\widetilde{\Gamma}^l\partial_rD^{kmn} + \frac{2}{3}D^{kmn}\partial_r\widetilde{\Gamma}^l - \frac{1}{6}\partial_r(\Theta^{lkmn} + A^{lkmn})\right)\\
&\quad + \frac{1}{3}\left(\widetilde{\Gamma}^l\partial^k(\Theta^{ijmn} + A^{ijmn}) - \partial^l(D^{rij}\partial_rD^{kmn}) + 2D^{mij}\partial^l\partial^k\widetilde{\Gamma}^n\right)\\
&\quad + \frac{1}{60}D_r{}^{mn}\left(B^{ijrs}D_s{}^{lk} - B^{ijlr}\widetilde{\Gamma}^k - 2\partial^rA^{lkij}\right) + \frac{1}{15}\left(B^{ijmr}\partial_rD^{nlk}\right.\\
&\quad\left.\left. + \frac{7}{6}\partial^l(B^{ijkr}D_r{}^{mn})\right)\right]\\
&k^8 : k_ik_jk_mk_nk_lk_kk_pk_q\left[\frac{1}{18}\left(\Theta^{ijmn}\Theta^{lkpq} + 4A^{ijmn}A^{lkpq} + 4\Theta^{ijmn}A^{lkpq}\right)\right.\\
&\quad + \frac{1}{2}\left(C^{ij}D^{pmn}D^{qlk} + D^{pij}D^{rmn}\partial_rD^{qlk}\right) + \frac{1}{3}\partial^p\left(\Theta^{ijmn} + A^{ijmn}\right)D^{qlk}\\
&\quad\left. + \left(\frac{1}{4}\widetilde{\Gamma}^i\widetilde{\Gamma}^j - \frac{1}{2}\partial^i\widetilde{\Gamma}^j\right)D^{pmn}D^{qlk} - \frac{1}{3}\left(\Theta^{ijmn} + 2A^{ijmn}\right)\widetilde{\Gamma}^pD^{qlk} - \frac{1}{12}B^{ijmr}D_r{}^{nl}D^{kpq}\right]\\
&k^{10} : k_ik_jk_mk_nk_lk_kk_pk_qk_rk_s\left[\frac{1}{6}\left(\widetilde{\Gamma}^rD^{sij}D^{pmn}D^{qlk} - \left(\Theta^{ijmn} + 2A^{ijmn}\right)D^{rpq}D^{slk}\right)\right]\\
&k^{12} : \frac{1}{24}k_ik_jk_mk_nk_lk_kk_pk_qk_rk_sk_uk_vD^{rij}D^{smn}D^{ulk}D^{vpq}
\end{aligned}
\tag{8.103}
$$

Chapter 9

Conclusions

In this thesis, we considered the coupling of non-relativistic field theories to curved backgrounds and their applications. We considered this in two parts. In the first part of the thesis, the curved background and the coupling of fields to them were determined through the localisation of the non-relativistic spacetime symmetries in flat space. This required a particular modification of the usual Poincaré gauge theory which we described as the Galilean Gauge theory (GGT). This was demonstrated to have a broad range of applicability through the localisation of the Galilean symmetries of both scalar and vector field theory models, as well as through the further inclusion of the anisotropic scale symmetry. The resulting general background were identified with a class of Newton-Cartan (NC) backgrounds through specific definitions of the vierbeins. In particular, the fields resulting from localising the Galilean symmetry led to the NC background, while the additional inclusion of dilations provided the scale covariant NC background. Apart from the modified coupling which fields have to these backgrounds, the backgrounds themselves were shown to have a particular degenerate metric structure with key differences from the ADM formulation of General Relativistic backgrounds. As a result, we expected that the dynamics of fields coupled to them would also have interesting characteristics. This was considered in the second part of the thesis. We first considered the formulation of fluids on the NC and scale covariant NC backgrounds, where we introduced a Weyl covariant formalism for the latter. We had also considered the effective field theory of a Hall droplet and demonstrated that the scale covariant background leads to an additional response function related to the expansion of the fluid. We had finally considered the trace and diffeomorphism anomalies of the Schrödinger field on the NC background. The modified structure of these anomalies are expected to play a role in the description of quantum fields and fluids on the NC background, as well as providing new features with regards to their RG flow. This is evident from the fact that the trace anomaly of the scalar field on the NC background in $2+1$ dimensions contains terms which satisfy both the a-theorem and the c-theorem.

In Chapter 2 a detailed discussion on the different approaches to Poincaré gauge theory was provided. The Lie algebraic, field theoretic approaches and the connection between these was been highlighted to better understand the Galilean Gauge theory (GGT) formulated in the following chapter. The field theoretic approach was demonstrated through the example of the complex Klein-Gordon field. The geometrical interpretation of the local Poincaré invariant action was discussed. In Chapter 3 we proposed a method of localising the global Galilean transformation of a general field theoretic model in order to construct spatial diffeomorphism invariant field theories. The localisation procedure in non-relativistic field theories required the separation of time and space. Local coordinate systems were considered to give local Galilean transformations a geometric interpretation. To restore the local invariance we first defined covariant derivatives with respect to the global coordinates and then transformed them to covariant derivatives with respect to the local coordinates. New fields were introduced in the process so that the local covariant derivatives transformed under local Galilean transformation as the ordinary derivatives did under global Galilean transformations. The localisation of the transformations also implied a change in the measure of integration for the matter action. We have shown that the measure can be determined appropriately by some functions of the fields introduced via localization. Substituting the local covariant derivatives and the measure, we obtained an action invariant under local Galilean transformations. The new fields along with their transformations were found to be useful in phenomenological model building in theoretical condensed matter physics such as in the theory of FQHE.

In Chapter 4 we considered a generic theory containing a free and interacting Schrödinger field with a gauge field. Imposing a constant time slice (by setting the time translation parameter to zero) the localisation procedure was shown to lead to a spatial diffeomorphism invariant action on general non-relativistic curved backgrounds. We also considered a model of an electron moving in two dimensional space whose dynamics was dictated by the Chern-Simons (CS) term. Contrary to certain results in the literature [55], the formulation of the action on curved backgrounds was shown to involve no problems up to certain boundary terms. The fields introduced through localisation admitted a geometric interpretation. In terms of these fields we constructed the Newton-Cartan (NC) background in Chapter 5. We demonstrated all basic properties of NC geometry, particularly those in relation to the dynamical description of matter fields coupled to it are satisfied following our construction. In Chapter 6 it was further demonstrated that the localisation procedure could be used to involve additional non-relativistic symmetries, such as anisotropic scale invariance. Inclusion of anisotropic scale symmetry introduced additional terms which modified the metricity conditions, the connection and the definition of the Riemann tensor of the scale covariant NC background in Chapter 6. These results motivated us to further investigate the dynamics of fields and fluids on these curved backgrounds.

In Chapter 7 we first recalled the formulation of fluids on the NC background, as well as the corresponding currents and constitutive relations for the case of ideal fluids. We then extended this formalism to account for fluids on the scale covariant NC background. We constructed a manifestly Weyl covariant framework within which such fluids may easily be investigated. This framework, along with the properties of the scale covariant Riemann tensor and Weyl tensor in Chapter 6 would provide the tools necessary to investigate non-relativistic fluids to higher orders in the derivative expansion. As an example, we considered the effective theory for a Quantum Hall fluid, described by the Landau-Ginzburg action. In particular we demonstrated that the effective action involved scale dependent terms in addition to the usual Berry phase and Wen-Zee terms. This additional term was shown to provide a response corresponding to the expansion of the fluid under deformations of the spatial metric. As explained, this provides only part of a much richer set of responses admitted by the scale covariant geometry. The analysis was entirely in the context of classical variations of the effective action. The presence of new response functions demonstrates that the one-loop results in the full quantum theory will be considerably more interesting.

To investigate quantum effects due to the NC background, particularly in relation to one-loop effects, we finally considered the trace and diffeomorphism anomalies of the Schrödinger field minimally coupled to the $2+1$ dimensional NC background in Chapter 8. This was performed within the Fujikawa approach, where all variations of the background fields were considered. The trace anomaly was shown to involve two pieces, with the form of the $3+1$ dimensional and the $1+1$ dimensional relativistic anomaly respectively. The diffeomorphism and trace anomalies further share a relation analogous to that of the $1+1$ dimensional scalar field. The presence of U(1) violating terms in the final result indicate the need to determine the U(1) gravitational anomaly. This would characterize the structure of non-relativistic anomalies, which as it already stands, is in sharp distinction with those of relativistic results. Unlike the relativistic case, the anomalies only arise in odd dimensions. The presence of anomaly terms which go like the relativistic results will also lead to interesting consequences. For the $3+1$ dimensional term, it was already demonstrated that part of it satisfies an a-theorem. In this thesis, we demonstrated that a part of the $1+1$ dimensional term satisfies a c-theorem.

The full treatment of local RG flows under marginal and non-marginal deformations and their relations with the correlation functions of the Schrödinger fields will be an important avenue to investigate in the future. This in particular might allow us to better understand the field A_μ of the NC background, its emergence in non-relativistic systems and its effect on the RG flow of correlators of the theory. This will be important in determining the critical points of condensed matter systems. One can also construct the effective gravitational action for the NC background using the trace anomaly result. In the relativistic case, the $1+1$ result can be integrated exactly. A similar property should hold for the NC trace anomaly in $2+1$ dimensions. Such a construction would result in the NC Liouville gravitational effective action. A long standing problem of the NC background has involved the construction of gravitational actions. Working with the result of the trace anomaly might allow us to better understand the dynamics of the NC background.

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